Cell Complexes for Arrangements with Group Actions[*]

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Abstract

For a real oriented hyperplane arrangement, we show that the corresponding Salvetti complex is homotopy equivalent to the complement of the complexified arrangement. This result was originally proved by M. Salvetti. Our proof follows the framework of a proof given by L. Paris and relies heavily on the notation of oriented matroids. We also show that homotopy equivalence is preserved when we quotient by the action of the corresponding reflection group. In particular, the Salvetti complex of the braid arrangement in ℓ dimensions modulo the action of the symmetric group is a cell complex which is homotopy equivalent to the space of unlabelled configurations of ℓ distinct points. Lastly, we describe a construction of the orbit complex from the dual complex for all finite reflection arrangements in dimension 2. This description yields an easy derivation of the so-called "braid relations" in the case of braid arrangement.

Acknowledgements

I would like to thank my advisor Dr. Michael Falk for providing me with a topic that completely captured my attention. I would also like to thank him for having an infinite amount of patience with me and allowing me to spend so much time in the chair in the corner of his office. Without him this thesis would not have been possible. I would also like to thank Dr. Janet McShane and Dr. Guenther Huck for taking the time to read my thesis and providing me with useful feedback.

Contents

List of Figures

Chapter 1

Posets and Cell Complexes

Cell Complexes A topological space E is called a *cell* of dimension n if it is homeomorphic to the unit disc, B^n. In this case we say that E is an n-cell. A space F of $\mathbb{R}^\ell$ is called an *open cell* of dimension n if it is homeomorphic to $\text{Int}(B^n)$. In this case we say that F is an open n-cell. We will refer to a 0-cell as a vertex.

Definition 1.1. We define a (finite) *cell complex* to be a Hausdorff topological space X together with a finite collection $F_0, \ldots, F_n$ of disjoint open cells in X such that

(1) $X = \bigcup_{k=0}^{n} F_k$.

(2) For each open n-cell F_k, there exists a continuous map

$$f_k : B^n \longrightarrow X$$

that maps $\text{Int}(B^n)$ homeomorphically onto F_k and carries $\text{Bd}(B^n)$ into a union of open cells, each of dimension less than n.

Note that what we are calling a cell complex is often referred to as a finite CW complex.

Example 1.2. The quotient space formed from B^ℓ by collapsing $\text{Bd}(B^\ell)$ to a point is homeomorphic to S^ℓ. Therefore, S^ℓ can be expressed as a cell complex having one open ℓ-cell and one 0-cell, and no other cells at all.

A cell complex X for which the maps f_k can be taken to be homeomorphisms is called a *regular cell complex*.

Example 1.3. The cell complex described in Example 1.2 is not a regular cell complex. However, we can give a regular cell complex structure to S^ℓ. Consider S^1. We can subdivide S^1 into two open 1-cells and two 0-cells. See Figure 1.1. The resulting cell complex is regular. Likewise, consider S^2. Take S^1 to be the equator of S^2 and let S^1 have the regular cell structure described above. Then S^2 will be a regular cell complex having two open 2-cells, two open 1-cells, and two 0-cells. Similarly, we can give S^ℓ a regular cell complex structure by subdividing S^ℓ into two open ℓ-cells, i.e., the open upper and lower hemispheres, and giving the equator the regular cell structure of $S^{\ell-1}$.
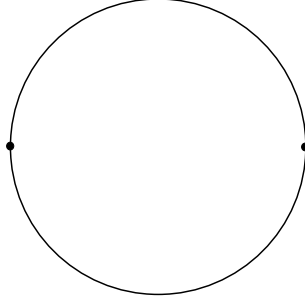
Figure 1.1: S^1 as a regular cell complex

We define the *k-skeleton* of X to be the subspace X^k of X that is the union of the open cells of X of dimension at most k. Note that if the highest dimensional cells of X are of dimension n, then $X = X^n$. Also, we have

$$X^0 \subset X^1 \subset \cdots \subset X^n = X.$$

Note that X^0 is the vertex set of X. Furthermore, note that $X^k - X^{k-1}$ is a disjoint union of open k-cells, possibly empty, for each k.

We sometimes refer to open cells as *faces*. Each open cell F of X is a face of X. We denote the set of faces of X by $\mathcal{F}(X)$, which is partially ordered via

$$F' \leq F \text{ iff } F' \subseteq \overline{F},$$

for all $F', F \in \mathcal{F}(X)$. We will refer to $\mathcal{F}(X)$ as the face poset of X.

Example 1.4. Let X be the regular cell complex given in Figure 1.2. Then the cells of X determine the face poset, $\mathcal{F}(X)$, given in Figure 1.3.

Figure 1.2: A regular cell complex

3

Figure 1.3: Face poset for a regular cell complex

In [2], Björner characterizes face posets of regular cell complexes. One consequence of his result is stated below without proof.

Proposition 1.5. *Let X be a regular cell complex. Then X is uniquely determined up to cellular homeomorphism by its face poset $\mathcal{F}(X)$.* $\qquad\square$

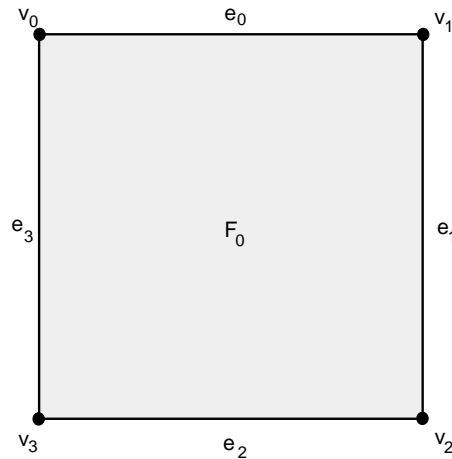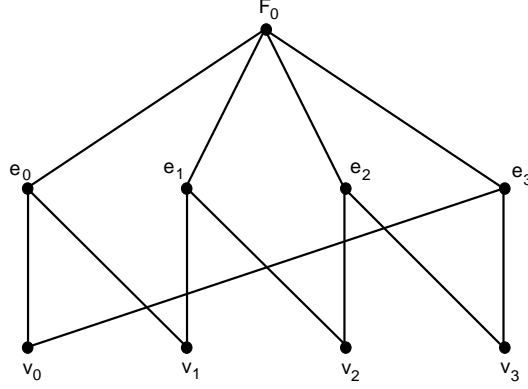**Concrete Simplicial Complexes**   Now, we will discuss a special class of regular cell complexes, called simplicial complexes. First, recall that a set $\{v_0, \ldots, v_n\}$ of points in a vector space V is said to be *geometrically independent* iff the vectors

$$v_1 - v_0, \ldots, v_n - v_0$$

are linearly independent. So, two distinct points in $\mathbb{R}^\ell$ form a geometrically independent set, as do three non-collinear points, four non-coplanar points, and so on.

If $\{v_0, \ldots, v_n\}$ is a geometrically independent set of points in $\mathbb{R}^\ell$, then we define the *concrete n-simplex* σ spanned by $v_0, \ldots, v_n$ to be the convex hull of $v_0, \ldots, v_n$, denoted

$$v_0 \vee \cdots \vee v_n.$$

The points $v_0, \ldots, v_n$ are called the vertices of σ.

In low dimensions, one can picture a concrete simplex easily. A 0-simplex is just a point, i.e., a vertex. A 1-simplex spanned by v_0 and v_1 is just the line segment joining v_0 and v_1. A 2-simplex spanned by v_0, v_1, and v_2 is the solid triangle having these three points as vertices. Similarly, a 3-simplex is a tetrahedron. For $n \geq 4$, an n-simplex is the n-dimensional analogue of a tetrahedron.

Any simplex spanned by a subset of $\{v_0, \ldots, v_n\}$, where $v_0, \ldots, v_n$ are the vertices of a simplex σ, is called a *face* of σ. The union of the faces of σ different from σ itself is called the *boundary* of σ and denoted $\mathrm{Bd}(\sigma)$. The *interior* of σ is defined by the equation

$$\mathrm{Int}(\sigma) = \sigma - \mathrm{Bd}(\sigma).$$

4

The interior of a simplex σ is called an *open simplex*. Note that if the vertices $v_0, \ldots, v_n$ span a simplex ϕ, then we will sometimes use

$$v_0 \vee \cdots \vee v_n$$

to denote the open simplex spanned by $v_0, \ldots, v_n$, making sure to clarify the meaning.

Definition 1.6. A *concrete simplicial complex* Δ in $\mathbb{R}^\ell$ is a collection of simplices in $\mathbb{R}^\ell$ such that

(i) Every face of a simplex of Δ is in Δ.

(ii) The intersection of any two simplexes of Δ is a face of each of them.

Example 1.7. The collection Δ_1 pictured in Figure 1.4(a), consisting of two 2-simplices with an edge in common, along their faces, is a simplicial complex. However, the collection Δ_2 pictured in Figure 1.4(b) is not a simplicial complex.

(a) Δ_1 (b) Δ_2
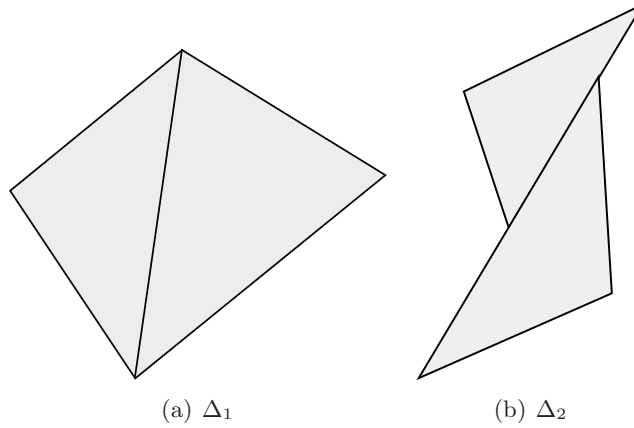
Figure 1.4: Collections of simplices

It is clear that a simplicial complex Δ is a regular cell complex, where the cells are the simplices. The collection of 0-simplices of Δ are the vertices of Δ, i.e., Δ^0. We will abuse notation and use Δ to refer to both the simplicial complex Δ and the subset of $\mathbb{R}^\ell$ that is the union of the simplices of Δ. Note that the same subset of $\mathbb{R}^\ell$ can have different simplicial structures.

Definition 1.8. If v is a vertex of a simplicial complex Δ, then the *star* of v in Δ, denoted $\mathrm{Star}(v)$, is the union of the interiors of those simplices of Δ that have v as a vertex.

Example 1.9. Let Δ be the simpicial complex given in Figure 1.5(a). Then the star of the vertex v_0 is shown in Figure 1.5(b).

Since a simplicial complex Δ is a regular cell complex, the simplices of Δ subdivide Δ into faces. Each simplex σ of Δ is a face of Δ. Again, we denote the set of faces of Δ by $\mathcal{F}(\Delta)$. The partial ordering on the face poset $\mathcal{F}(\Delta)$ is the same as before. Furthermore, a simplex σ in $\mathcal{F}(\Delta)$ is uniquely determined by its set of vertices.

(a) Δ (b) Star(v_0)

Figure 1.5: The star of a vertex

Abstract Simplicial Complexes The partial ordering on simplices of a simplicial complex together with the last remark above leads us to the following, more general, definition of simplicial complex.
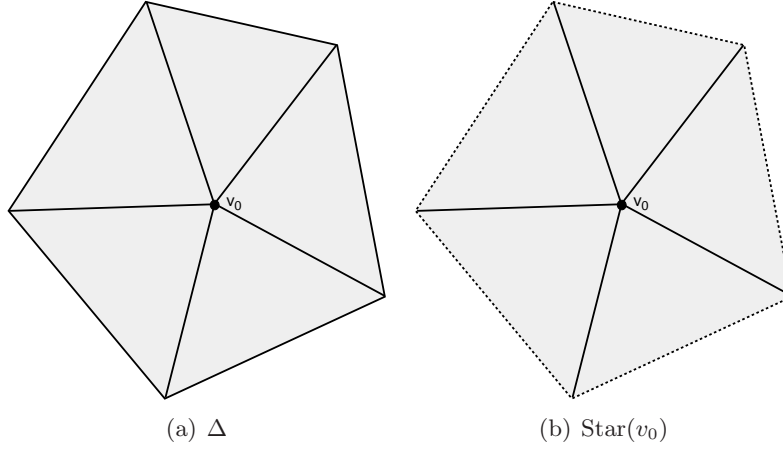
Definition 1.10. An *abstract simplicial complex* is a collection $\mathcal{S}$ of finite nonempty sets, such that if A is an element of $\mathcal{S}$, then so is every nonempty subset of A.

The element A of $\mathcal{S}$ is called a simplex of $\mathcal{S}$. If A consists of $n+1$ elements, then A is an n-simplex, and is said to have dimension n. Each nonempty subset of A is called a face of A. By hypothesis, each face of A is also a simplex. The vertex set of $\mathcal{S}$ is the union of all the 0-simplices of $\mathcal{S}$, where the 0-simplices are the singletons of $\mathcal{S}$.

Every abstract simplicial complex is partially ordered by set inclusion. We say that that two abstract simplicial complexes $\mathcal{S}$ and $\mathcal{T}$ are isomorphic if they are isomorphic as posets.

Definition 1.11. Let Δ be a concrete simplicial complex. Define $V(\Delta)$ to be the collection of all subsets $\{v_0, \ldots, v_n\}$ of Δ^0 such that the vertices $v_0, \ldots, v_n$ span an n-simplex of Δ. The collection $V(\Delta)$ is called the *vertex scheme* of Δ, and is partially ordered by inclusion.

It should be clear that the vertex scheme for a concrete simplicial complex Δ satisfies the criteria for being an abstract simplicial complex. That is, $V(\Delta)$ is an abstract simplicial complex.

Example 1.12. Let Δ be the concrete simplicial complex of Figure 1.6(a). Then the vertex scheme $V(\Delta)$ is given in Figure 1.6(b).

Since $\{v_0, \ldots, v_n\}$ is an element of $V(\Delta)$ only when $v_0, \ldots, v_n$ span a simplex of Δ, it should be clear that $\mathcal{F}(\Delta)$ and $V(\Delta)$ are isomorphic as posets. For this reason, we will consider $\mathcal{F}(\Delta)$ to be an abstract simplicial complex.

Not only is $\mathcal{F}(\Delta)$ an example of an abstract simplicial complex, it is the crucial example, as the following proposition shows.
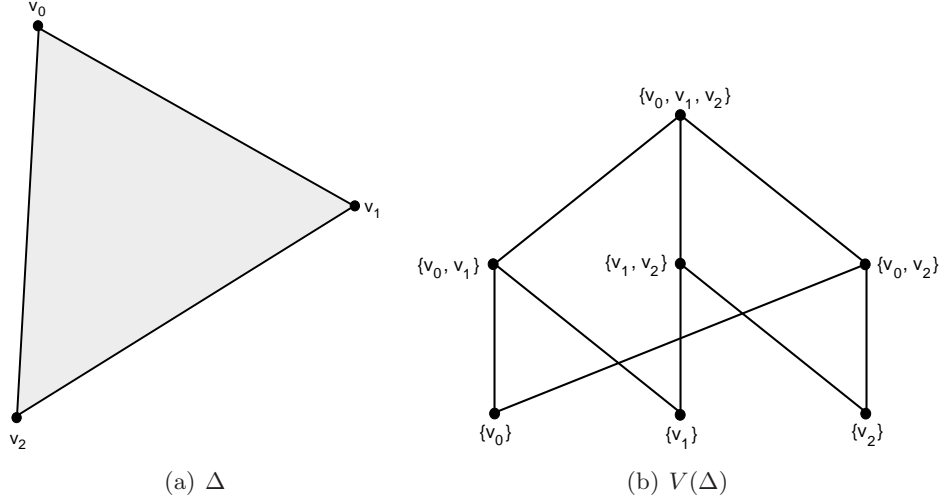
6

Figure 1.6: A simplicial complex and its vertex scheme

Proposition 1.13. *Every abstract simplicial complex $\mathcal{S}$ is isomorphic to the vertex scheme of some concrete simplicial complex Δ. That is, $\mathcal{S}$ is isomorphic to the face poset of some concrete simplicial complex Δ. Furthermore, Δ is uniquely determined up to simplicial homeomorphism.*

Proof. Say that $v_0, \ldots, v_n$ are the vertices of $\mathcal{S}$. We can easily form a realization of $\mathcal{S}$ in $\mathbb{R}^{n+1}$ in the following way. Let e_k denote the kth standard basis vector in $\mathbb{R}^{n+1}$. Form a concrete simplex spanned by a subset of $\{e_0, \ldots, e_n\}$ exactly when the corresponding subset of $\{v_0, \ldots, v_n\}$ is an abstract simplex. The resulting concrete simplicial complex Δ is a realization of $\mathcal{S}$. That is, $V(\Delta)$ is isomorphic to $\mathcal{S}$. $\qquad\square$

Note that Δ in the proof above is a subcomplex of the standard n-simplex in $\mathbb{R}^{n+1}$.

Definition 1.14. If the abstract simplicial complex $\mathcal{S}$ is isomorphic to the face poset of the concrete simplicial complex Δ, then we call Δ a realization of $\mathcal{S}$. Δ is unique up to simplicial homeomorphism. We will denote a realization of $\mathcal{S}$ by $|\mathcal{S}|$.

The Order Complex   Given any poset P, we can form its *order complex*, denoted $\operatorname{Ord}(P)$. This is an abstract simplicial complex whose vertices are the elements of P and whose simplices are the chains

$$p_0 < \cdots < p_n$$

in P. We will use the notation

$$\Omega = (\omega_0 < \cdots < \omega_n)$$

to denote a simplex of the order complex. Note that $\operatorname{Ord}(P)$ is an abstract simplicial complex since any subchain of a chain is also a chain. Also, note that the realization $|\operatorname{Ord}(P)|$ is a concrete simplicial complex.

In the special case where $\mathcal{F}(X)$ is the face poset for a regular cell complex X, the order complex of $\mathcal{F}(X)$ is called the *barycentric subdivision* of X, denoted X'. We can form a realization of X' in the following way.

Let F be a cell of a regular cell complex X. Then $\overline{F}$ is homeomorphic to B^{ℓ} for some ℓ. Also, we can give B^{ℓ} a regular cell structure which is isomorphic to the regular cell structure of $\overline{F}$. Let $B_{F'}$ denote the face of B^{ℓ} which corresponds to the face F' of $\overline{F}$. For every open face F' of $\overline{F}$, select a point $x(F') \in F'$, called the *barycenter* of F'. Then each point $x(F')$ determines a point $x(B_{F'}) \in B_{F'}$, since there is a homeomorphism between F' and $B_{F'}$. Whenever

$$F_0 < \cdots < F_n$$

is a chain of faces of $\overline{F}$ form the simplex

$$\phi_B = x(B_{F_0}) \vee \cdots \vee x(B_{F_n})$$

on B^{ℓ}. Recall that ϕ_B is the convex hull of the vertices $x(B_{F_0}), \ldots, x(B_{F_n})$ on B^{ℓ}. Since there is homeomorphism between B^{ℓ} and $\overline{F}$, each simplex ϕ_B of B^{ℓ} determines the simplex ϕ of $\overline{F}$ having $x(F_0), \ldots, x(F_n)$ as its vertices. Even though ϕ may not be the convex hull of the vertices $x(F_0), \ldots, x(F_n)$, we will still let

$$x(F_0) \vee \cdots \vee x(F_n)$$

denote the simplex ϕ. The resulting cell complex is homeomorphic to a concrete simplicial complex in $\mathbb{R}^{\ell}$.

Note that we can form the barycentric subdivision of any regular cell complex over and over again. Each subsequent subdivision creates more simplicial cells.

Example 1.15. Consider the regular cell complex X indicated in Figure 1.7(a). Its first barycentric subdivision, X', is pictured in Figure 1.7(b).
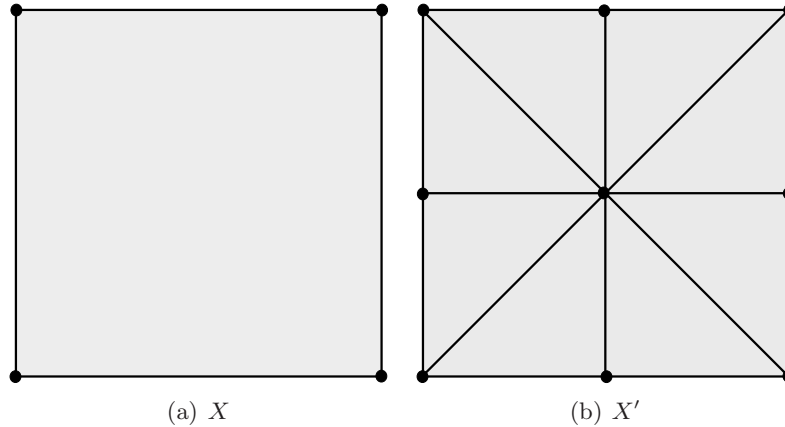

(a) X (b) X'

Figure 1.7: Barycentric subdivisions of a regular cell complex

Barycentric subdivision of a regular cell complex X leaves all of its topological properties intact. Hence we have the following proposition.

Proposition 1.16. *If X is a regular cell complex, then X and X' are homeomorphic as topological spaces.* $\square$

We will now define a special example of an abstract simplicial complex which we will make use of in later chapters.

Definition 1.17. Let $\mathcal{U}$ be a collection of subsets of $\mathbb{R}^\ell$. We define an abstract simplicial complex called the *nerve* of $\mathcal{U}$, denoted $N(\mathcal{U})$. Its vertices are the elements of $\mathcal{U}$ and its simplices are the finite subcollections $\{U_0, \ldots, U_n\}$ such that

$$U_0 \cap \cdots \cap U_n \neq \emptyset.$$

Note that $N(\mathcal{U})$ is an abstract simplicial complex, since if

$$U_0 \cap \cdots \cap U_n \neq \emptyset,$$

then any subcollection of $\{U_0, \ldots, U_n\}$ also has nonempty intersection.

The following theorem about the nerve of open covers is called the Nerve Theorem. We will not make use of this theorem until Chapter 5. For a proof, see [3].

Theorem 1.18. *Let $\mathcal{U}$ be an open cover of contractible sets of a subset A of $\mathbb{R}^\ell$ (or $\mathbb{C}^\ell$). Then $|N(\mathcal{U})|$ has the same homotopy type as A.* $\square$

For a closer look at this development of regular cell complexes and simplicial complexes see [8].

Chapter 2

Hyperplane Arrangements

By a *hyperplane* H in $\mathbb{R}^\ell$ we mean a linear subspace $H \subseteq \mathbb{R}^\ell$ of codimension 1. That is, we are assuming a hyperplane H contains the origin $\{0\}$. Each hyperplane H is given by a linear function $\alpha(x)$, so that

$$H = \{x \in \mathbb{R}^\ell : \alpha(x) = 0\}.$$

The choice of α determines a "positive" and a "negative" side of H. The positive side of H is given by

$$H^+ = \{x \in \mathbb{R}^\ell : \alpha(x) > 0\}$$

and the negative side of H is given by

$$H^- = \{x \in \mathbb{R}^\ell : \alpha(x) < 0\}.$$

It is clear that an alternative choice of a linear function α for H is simply the choice of a positive side, up to positive scalar multiple. When the positive side of a hyperplane H has been specified, we call H an *oriented hyperplane*.

Definition 2.1. A *real hyperplane arrangement* $\mathcal{A}$ in $\mathbb{R}^\ell$ is a finite collection of hyperplanes $\{H_1, \ldots, H_n\}$ in $\mathbb{R}^\ell$. Often, we will refer to a real hyperplane arrangement simply as a hyperplane arrangement, unless the context is unclear.

If the positive side of each hyperplane H_i in $\mathcal{A}$ has been chosen, then we call $\mathcal{A}$ an *oriented hyperplane arrangement*. We say that $\mathcal{A}$ is *essential* if

$$\bigcap_{H \in \mathcal{A}} H = \{0\}.$$

Example 2.2. Figure 2.1 depicts an essential hyperplane arrangement $\mathcal{A}$ consisting of three hyperplanes H_1, H_2, and H_3 in $\mathbb{R}^2$. The arrows on each hyperplane indicate the positive side.

The hyperplanes of $\mathcal{A}$ subdivide $\mathbb{R}^\ell$ into *faces*. We denote the set of all faces by $\mathcal{F}(\mathcal{A})$, which is partially ordered via

$$F' \leq F \text{ iff } F' \subseteq \overline{F},$$

for all $F', F \in \mathcal{F}(\mathcal{A})$. We will refer to $\mathcal{F}(\mathcal{A})$ as the face poset for $\mathcal{A}$. A face $C \in \mathcal{F}(\mathcal{A})$ is called a *chamber* if C is a codimension 0 face of $\mathcal{A}$. We denote the set of all chambers of $\mathcal{A}$ by $\mathcal{C}(\mathcal{A})$. Note
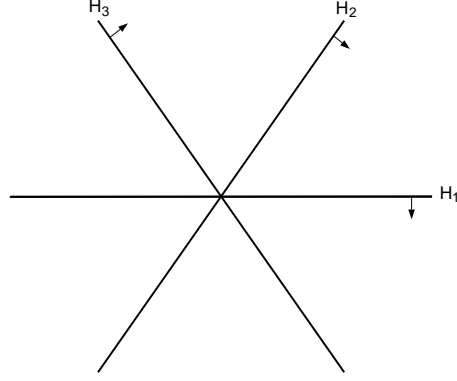
Figure 2.1: An essential hyperplane arrangement

that $\mathcal{C}(\mathcal{A})$ consists exactly of the maximal elements of the face poset. Given a chamber C of $\mathcal{A}$, its *walls* are defined to be the hyperplanes H_i of $\mathcal{A}$ such that $\overline{C} \cap H_i \neq \emptyset$. Two chambers C and D are called *adjacent* if they have a common wall.

Example 2.3. Let $\mathcal{A}$ be the hyperplane arrangement of Figure 2.2. Then

$$\mathcal{C}(\mathcal{A}) = \{C_0, C_1, C_2, C_3, C_4, C_5\}.$$

The walls of C_0 are the hyperplanes H_1 and H_2. The face poset for $\mathcal{A}$ is given in Figure 2.3.
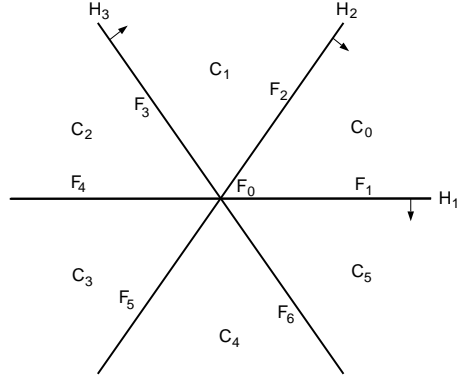
Figure 2.2: Faces of a hyperplane arrangement

A *subarrangement* of $\mathcal{A}$ is any subcollection of hyperplanes of $\mathcal{A}$. For example, the walls of any chamber C of $\mathcal{A}$ form a subarrangement. Also, for all $F \in \mathcal{F}(\mathcal{A})$, we can form the subarrangement

$$\mathcal{A}_F = \{H \in \mathcal{A} : F \subseteq H\}.$$

Note that $\mathcal{A}_F = \mathcal{A}_{F'}$ iff $|F| = |F'|$, where $|F|$ and $|F'|$ are the linear subspaces spanned by F and F', respectively. Also, note that if $C \in \mathcal{C}(\mathcal{A})$, then $\mathcal{A}_C = \emptyset$.

If $C \in \mathcal{C}(\mathcal{A})$ and $F \in \mathcal{F}(\mathcal{A})$, then we denote by C_F the unique chamber of $\mathcal{A}_F$ containing C. Note that, since $\mathcal{A}_C = \emptyset$ for $C \in \mathcal{C}(\mathcal{A})$, $C'_C = \mathbb{R}^\ell$, for any two chambers C' and C of $\mathcal{A}$.
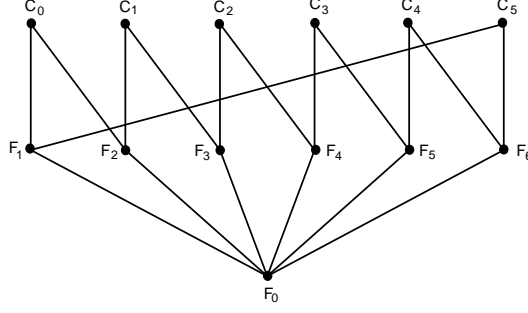
Figure 2.3: Face poset for a hyperplane arrangement

Example 2.4. Figure 2.4(b) depicts the subarrangement $\mathcal{A}_F$, where $\mathcal{A}$ is the hyperplane arrangement of Figure 2.4(a). Figure 2.4(b) also shows the unique chamber C_F.

(a) $\mathcal{A}$ (b) $\mathcal{A}_F$
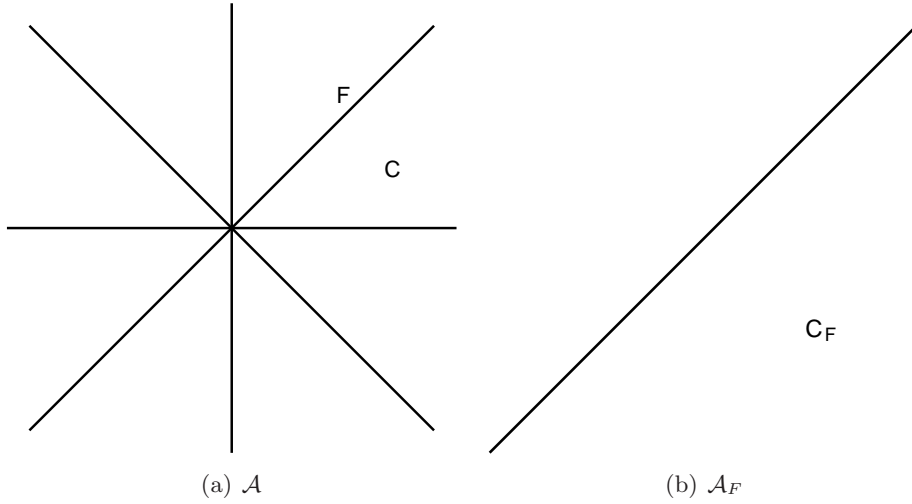
Figure 2.4: A subarrangement

If $\mathcal{A}$ is a real hyperplane arrangement, then

$$\mathcal{A}_{\mathbb{C}} = \{H + iH : H \in \mathcal{A}\} \subseteq \mathbb{C}^{\ell}$$

is called the *complexification of* $\mathcal{A}$. The defining equations for each complex hyperplane $H + iH$ are the same as for the real hyperplane H, except now the independent variables are complex numbers. Note that a real arrangement $\mathcal{A}$ is essential iff its complexification is essential. Also note that $x + iy \in H + iH$ iff $x \in H$ and $y \in H$.

If $\mathcal{A}$ is a real hyperplane arrangement, then we will denote the *complement of the complexification of* $\mathcal{A}$ via

$$M(\mathcal{A}) = \mathbb{C}^{\ell} - \bigcup_{H \in \mathcal{A}} (H + iH).$$

That is, $M(\mathcal{A})$ consists of all points $x + iy \in \mathbb{C}^\ell$ where both x and y are not contained in the same hyperplane H.

Every essential arrangement $\mathcal{A}$ in $\mathbb{R}^\ell$ determines a (regular) cellular decomposition of $S^{\ell-1}$, denoted $S_\mathcal{A}$. A face $F \neq \{0\}$ of $\mathcal{A}$ corresponds to the open cell $\Delta(F) = F \cap S^{\ell-1}$, and every open cell of this decomposition of $S^{\ell-1}$ has that form.

This cellular decomposition of $S^{\ell-1}$ determines a simplicial decomposition of $S^{\ell-1}$, i.e., the barycentric subdivision. For every $F \neq \{0\}$ of $\mathcal{A}$ we fix a point $x(F) \in \Delta(F)$. A chain

$$\{0\} \neq F_0 < \cdots < F_n$$

of faces of $\mathcal{A}$ determines an open simplex

$$\phi = x(F_0) \vee \cdots \vee x(F_n)$$

having $x(F_0), \ldots, x(F_n)$ as vertices, and every simplex of this simplicial decomposition of $S^{\ell-1}$ has that form. Let $S'_\mathcal{A}$ denote the simplicial decompostion of $S^{\ell-1}$ determined by $\mathcal{A}$.

The simplicial decomposition of $S^{\ell-1}$ determinies a simplicial decomposition of B^ℓ. We add the vertex $x(\{0\}) = 0$ to the set of vertices for $S^{\ell-1}$. A chain

$$F_0 < \cdots < F_n$$

of faces of $\mathcal{A}$ determines an open simplex

$$\phi = x(F_0) \vee \cdots \vee x(F_n)$$

having $x(F_0), \ldots, x(F_n)$ as vertices, and every simplex of this simplicial decomposition of B^ℓ has that form. Let $B'_\mathcal{A}$ denote the simplicial decomposition of B^ℓ determined by $\mathcal{A}$.

Example 2.5. Let $\mathcal{A}$ be the hyperplane arrangement of Figure 2.1. Then the simplicial decomposition, $S'_\mathcal{A}$, is shown in Figure 2.5(a). By adding the vertex $x(\{0\})$ to $S'_\mathcal{A}$, we get the simplicial decomposition, $B'_\mathcal{A}$, shown in Figure 2.5(b).

Note that, if $F_0 \neq \{0\}$, then
$$\phi = x(F_0) \vee \cdots \vee x(F_n) \subseteq S^{\ell-1}.$$

If ϕ is an open cell of $S^{\ell-1}$, then the *cone of* ϕ is

$$K(\phi) = \{\lambda x : x \in \phi \text{ and } \lambda > 0\}.$$

Lemma 2.6. *If*
$$\phi = x(F_0) \vee \cdots \vee x(F_n)$$

is an open simplex of $S^{\ell-1}$ with
$$\{0\} \neq F_0 < \cdots < F_n,$$

then $K(\phi) \subseteq F_n$.

(a) $S'_{\mathcal{A}}$ (b) $B'_{\mathcal{A}}$
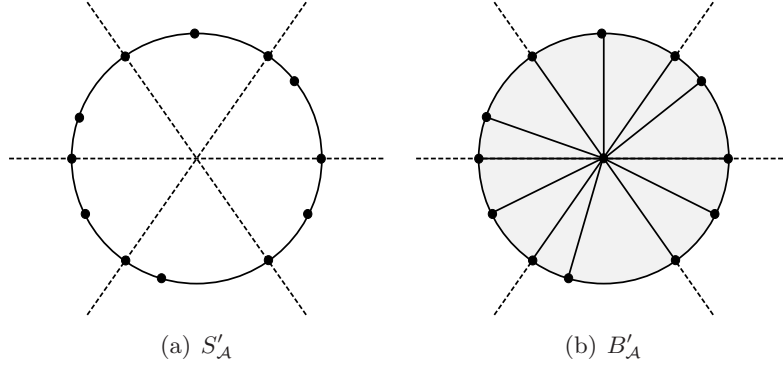
Figure 2.5: Simplicial decomposition of S^1 and B^2

Proof. Let $y \in K(\phi)$. Then $y = \lambda_0 x$ for some $x \in \phi$ and some $\lambda_0 > 0$. Since $F_0 \leq \cdots \leq F_n$, $x(F_0), \ldots, x(F_n) \in \overline{F_n}$. Then the open convex hull of $x(F_0), \ldots, x(F_n)$ is contained in F_n, since F_n is convex. Hence $\phi \subseteq F_n$. So, $x \in F_n$. But $\lambda x \in F_n$ for every $\lambda > 0$, since F_n is a sector. Hence $y \in F_n$. Therefore, $K(\phi) \subseteq F_n$. $\qquad\square$

Note that the family

$$\{K(\phi) : \phi \text{ a simplex of } S^{\ell-1}\}$$

is a partition of $\mathbb{R}^\ell - \{0\}$.

Chapter 3

Arrangements with Group Actions

There are many special types of hyperplane arrangements. In this chapter, we will focus our discussion on arrangements which support the action of finite reflection groups. We will begin by introducing some necessary definitions and terminology.

Let H be a hyperplane in $\mathbb{R}^\ell$. The *reflection* with respect to H is the linear transformation

$$s_H : \mathbb{R}^\ell \longrightarrow \mathbb{R}^\ell$$

which is the identity on H and is multiplication by -1 on the orthogonal complement $H^\perp$ of H.

Definition 3.1. A *finite reflection group* is a finite group W of orthogonal linear transformations of $\mathbb{R}^\ell$ generated by reflections.

If W is a finite reflection group, then the corresponding hyperplane arrangement $\mathcal{A}_W$ consists of all the hyperplanes H where s_H is a reflection in W. We will usually suppress the subscript W when referring to the hyperplane arrangement corresponding to the finite reflection group W. In this case, we will call $\mathcal{A}$ a *reflection arrangement*.

For a complete proof of the following theorem, see [5] or [7].

Theorem 3.2. *If $\mathcal{A}$ is a reflection arrangement with reflection group W, then W permutes the hyperplanes of $\mathcal{A}$.*

Proof. One easily verifies that $s_{wH} = w s_H w^{-1}$ when $s_H \in W$ and $w \in W$. This implies that W permutes the hyperplanes of $\mathcal{A}$. $\square$

Since each reflection $s_H \in W$ is linear and every face of $\mathcal{A}$ is an intersection of half-spaces and hyperplanes of $\mathcal{A}$, we have the following corollary.

Corollary 3.3. *If $\mathcal{A}$ is a reflection arrangement with reflection group W, then W permutes the faces of $\mathcal{A}$. Furthermore, W preserves the face relation determined by $\mathcal{A}$.* $\square$

The following lemma provides us with a useful way of thinking about reflections.

Lemma 3.4. *If s_{H_i} and s_{H_j} are reflections over the hyperplanes H_i and H_j in $\mathbb{R}^\ell$, respectively. Then $s_{H_i} s_{H_j} \in W$ and $s_{H_i} s_{H_j}$ is a rotation about $H_i \cap H_j$ of twice the angle between H_i and H_j.* $\square$

The requirement that W be finite is very important. Suppose, for instance, that W contains the reflections s_{H_1} and s_{H_2}. Since W is finite, the rotation $s_{H_1} s_{H_2}$ has finite order. Then the angle between H_1 and H_2 must be a rational multiple of π. In higher dimensions, the restrictions on the angles between any two hyperplanes is even more severe. In general, we have the following result.

Proposition 3.5. *Let $\mathcal{A}$ be an essential reflection arrangement in $\mathbb{R}^\ell$ with reflection group W. When $\ell = 2$, W is a finite reflection group iff the angle between any pair of hyperplanes H_i and H_j of $\mathcal{A}$ is a rational multiple of π. When $\ell > 2$, W is an irreducible finite reflection group only if the angle between any pair of hyperplanes H_i and H_j of $\mathcal{A}$ is $\pi/2$, $\pi/3$, $\pi/4$, $\pi/5$, or $\pi/6$.* $\qquad\square$

Here, irreducible means irreducible as representations of abstract groups. Finite reflection groups have been completely classified up to isomorphism. Below, we list the irreducible finite reflection groups briefly with little explanation. Also, the hyperplane arrangements associated to each reflection group are assumed to be essential. All other finite reflection groups can be obtained from these by taking direct sums and, possibly, adding an extra summand on which the group acts trivially.

- *Type A_n ($n \geq 1$):* Here W is the symmetric group on $n + 1$ letters, acting on a certain n-dimensional subspace of $\mathbb{R}^{n+1}$ by permuting the coordinates. This group is the group of symmetries of a regular n-simplex, and it can also be described as the Weyl group of the root system of type A_n.

- *Type B_n ($n \geq 2$):* This is the group W of signed permutations acting on $\mathbb{R}^n$. The group W is the group of symmetries of the n-cube $[-1, 1]^n$; it is also the Weyl group of the root system of type B_n.

- *Type D_n ($n \geq 4$):* This group W is the Weyl group of a root system, called the root system of type D_n, but it does not correspond to any regular solid. It is the group of signed permutations with an even number of sign changes. It also happens to be a subgroup of index 2 of the reflection group of type B_n.

- *Type E_n ($n = 6, 7, 8$):* This is the Weyl group of the root system of the same name. It does not correspond to any regular solid.

- *Type F_4:* This is the Weyl group of the root system of the same name; it is also the group of symmetries of a certain self-dual 24-sided regular solid in $\mathbb{R}^4$ whose 1-codimensional faces are solid octahedra.

- *Type H_n ($n = 3, 4$):* This does not correspond to any root system, but it is the symmetry group of a regular solid X. When $n = 3$, X is the dodecahedron or, dually, the icosahedron. When $n = 4$, X is a 120-sided solid in $\mathbb{R}^4$ with dodecahedral faces or, dually, a 600-sided solid with tetrahedral faces.

- *Type $I_2(m)$ ($m = 5$ or $m \geq 7$):* The group W is the dihedral group of order $2m$. It is the symmetry group of a regular m-gon, but it does not correspond to any root system.

Example 3.6. Figure 3.1 shows the square with its lines of symmetry. The four lines of symmetry form the hyperplane arrangement $\mathcal{A}$, which has the corresponding finite reflection group of type B_2. The finite reflection group consists of all distinct products of the reflections $s_{H_1}, s_{H_2}, s_{H_3}$, and s_{H_4}. Notice that for any pair of chambers of $\mathcal{A}$, there exists a product of reflections which maps one chamber to the other. That is, B_2 acts transitively on the chambers of $\mathcal{A}$.
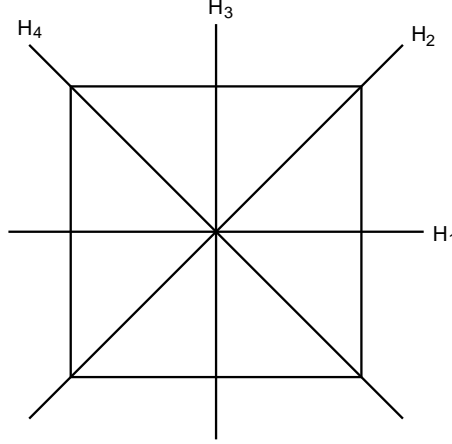
Figure 3.1: A finite reflection arrangement of type B_2

Now, assume that $\mathcal{A}$ is an essential reflection arrangement in $\mathbb{R}^\ell$ with reflection group W. Anytime we fix a chamber C_0 of $\mathcal{A}$, we will refer to C_0 as the *base chamber* for $\mathcal{A}$. Using this notion of base chamber, we will list a series of facts about finite reflection groups without proof.

Lemma 3.7. *Let $\mathcal{A}$ be an essential reflection arrangement in $\mathbb{R}^\ell$ with reflection group W. Suppose that C_0 is the base chamber of $\mathcal{A}$ with walls $H_1, \ldots, H_n$. Then W is generated by the reflections $s_{H_1}, \ldots, s_{H_n}$.* $\qquad\square$

Lemma 3.8. *Let $\mathcal{A}$ be an essential reflection arrangement in $\mathbb{R}^\ell$ with corresponding reflection group W. Then the action of W is transitive and free on the set of chambers. In particular, the number of chambers is equal to the order of W.* $\qquad\square$

Recall that an action is *free* if point stabilizers are trivial. To prove that W acts freely, one shows that any $s \in W$ which fixes a closed chamber $\overline{C}$ setwise, also fixes $\overline{C}$ pointwise, and hence must be the identity.

Lemma 3.8 also implies that there is a one-to-one correspondence between the chambers of $\mathcal{A}$ and the elements of W. That is, we can identify chambers with elements of W. If C_0 is the base chamber of $\mathcal{A}$, then identify C_0 with the identity element of W. Then for a chamber $C \neq C_0$ of $\mathcal{A}$, identify C with the unique element $w \in W$ such that $wC_0 = C$. The element w exists because W is transitive and is unique since W acts freely on chambers. Note that we can write w as a product of reflections $s_{H_1}, \ldots, s_{H_n}$, where $H_1, \ldots, H_n$ are the walls of C_0, according to Lemma 3.7.

Lemma 3.9. *Let $\mathcal{A}$ be an essential reflection arrangement in $\mathbb{R}^\ell$ with reflection group W. Suppose that C_0 is the base chamber of $\mathcal{A}$ with walls $H_1, \ldots, H_n$, so that $S = \{s_{H_1}, \ldots, s_{H_n}\}$ is the set of reflections with respect to the walls of C_0. Then $\overline{C_0}$ is a set of representatives for the W-orbits in $\mathbb{R}^\ell$. Moreover, the stabilizer W_x of a point $x \in \overline{C_0}$ is the subgroup generated by $S_x = \{s \in S : sx = x\}$. In particular, W_x fixes every point of $\overline{F}$, where $F \leq C_0$ is the smallest face containing x.* $\qquad\square$

Lemma 3.10. *Let $\mathcal{A}$ be an essential reflection arrangement in $\mathbb{R}^\ell$ with reflection group W. Suppose that C_0 is the base chamber of $\mathcal{A}$ with walls $H_1, \ldots, H_n$. Then each face F of $\mathcal{A}$ is congruent to a*

17

unique face F_0 of C_0 under the action of W. Furthermore, if $w \in W$ with $wC_0 = C$ and $F_0 \le C_0$, then $wF_0 = F \le C$. $\quad\square$

Using the above series of lemmas, we can show that the simplicial decomposition of $S^{\ell-1}$ determined by $\mathcal{A}$ can be constructed to be invariant under the action of W.

Theorem 3.11 (Salvetti [10]). *If $\mathcal{A}$ is an essential reflection arrangement in $\mathbb{R}^\ell$ with reflection group W, then we can construct $S'_{\mathcal{A}}$ to be W-invariant.*

Proof. Let $\mathcal{A}$ be an essential reflection arrangement in $\mathbb{R}^\ell$ with reflection group W. First, fix a base chamber C_0 of $\mathcal{A}$. Suppose that $F_{01}, \ldots, F_{0n}$ are the nonzero faces of C_0. Now, pick a point $x(C_0) \in C_0 \cap S^{\ell-1}$ and a point $x(F_{0j}) \in F_{0j} \cap S^{\ell-1}$ for each j. For each chamber $C_i \ne C_0$ of $\mathcal{A}$ with nonzero faces $F_{i1}, \ldots, F_{in}$, suppose that $s_i \in W$ is identified with C_i. That is,

$$C_i = s_i C_0$$

for each i. We can choose s_i uniquely according to the remark following Lemma 3.8. Note that we know that each C_i has the same number of faces as C_0 by Lemma 3.10. Now, set

$$x(C_i) = s_i x(C_0)$$

for each i. Also by Lemma 3.10, each face of C_i is congruent to a unique face of C_0. Without loss of generality, assume that F_{ij} is congruent to F_{0j} for each j. Then we can set

$$x(F_{ij}) = s_i x(F_{0j})$$

for each j, according to Lemma 3.10. It is clear that each

$$x(F_{ij}) \in F_{ij} \cap S^{\ell-1}.$$

Now, for every chain of faces

$$F_1 < \cdots < F_n,$$

form the simplex

$$x(F_1) \vee \cdots \vee x(F_n)$$

on $S^{\ell-1}$. It is clear that W is invariant on simplices of this form since each $s \in W$ is linear and W preserves order. With the simplicial decomposition of $S^{\ell-1}$ constructed in this way, it is clear that $S'_{\mathcal{A}}$ is W-invariant. $\quad\square$

Note that if we include the point $x(\{0\})$ in our decomposition, then we can also construct $B'_{\mathcal{A}}$ to be W-invariant.

Example 3.12. Consider the reflection arrangement $\mathcal{A}$ given in Figure 3.2. Let W be the reflection group of $\mathcal{A}$ and let C_0 be the base chamber. For the faces F_1 and F_2 of C_0 we fix the points $x(F_1) \in F_1 \cap S^1$ and $x(F_2) \in F_2 \cap S^1$. See Figure 3.3. In this case, we don't have much choice as to what $x(F_1)$ and $x(F_2)$ are. Next, pick $x(C_0) \in C_0 \cap S^1$. The rest of the vertices of $S'_{\mathcal{A}}$ are chosen by looking at the orbits of the points $x(F_1), x(F_2)$, and $x(C_0)$, under combinations of reflections over the walls of C_0. It should be clear from Figure 3.3 that the indicated points are what we get. The resulting simplicial decomposition of S^1 will be invariant under the action of W.
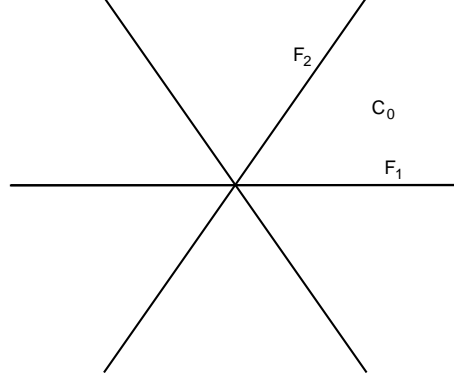
Figure 3.2: A hyperplane arrangement
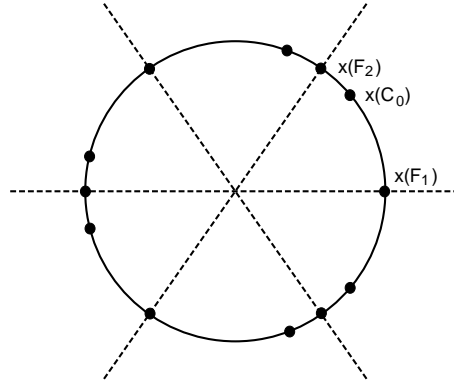
Figure 3.3: W-invariant $S'_{\mathcal{A}}$

The Braid Arrangement One class of reflection arrangements which are of particular interest to us are called the *braid arrangements*. We denote the braid arrangement in $\mathbb{R}^{\ell}$ by $\mathcal{A}_{\ell-1}$. For each dimension ℓ, we define

$$\mathcal{A}_{\ell-1} = \{H_{ij} : 1 \leq i < j \leq \ell\} \subseteq \mathbb{R}^{\ell}$$

where $H_{ij} = \{x \in \mathbb{R}^{\ell} : x_i = x_j\}$. We orient each H_{ij}, so that $x_i > x_j$ defines the positive side of H_{ij} for $i < j$. So, each H_{ij} is positive on the side containing the positive x_i axis.

Clearly, each $\mathcal{A}_{\ell-1}$ is not essential, since

$$\bigcap \mathcal{A}_{\ell-1} = \{x \in \mathbb{R}^{\ell} : x_1 = \cdots = x_{\ell}\} \neq \{0\}.$$

However, we can "make" $\mathcal{A}_{\ell-1}$ essential and still preserve the face poset for $\mathcal{A}_{\ell-1}$. Take the plane defined by

$$x_1 + \cdots + x_{\ell} = 0$$

perpendicular to the line

$$\{x_1 = \cdots = x_{\ell}\}$$

19

and then intersect the hyperplanes of $\mathcal{A}_{\ell-1}$ with that plane. The "new" arrangement will be isomorphic to an arrangement in $\mathbb{R}^{\ell-1}$ and will preserve the structure of the face poset for $\mathcal{A}_{\ell-1}$.

Example 3.13. Figure 3.4 represents the braid arrangement $\mathcal{A}_2$. Note that this is not $\mathcal{A}_2$, but rather a projection of $\mathcal{A}_2$ onto the plane defined by

$$x_1 + x_2 + x_3 = 0$$

perpendicular to the line

$$\{x_1 = x_2 = x_3\}.$$

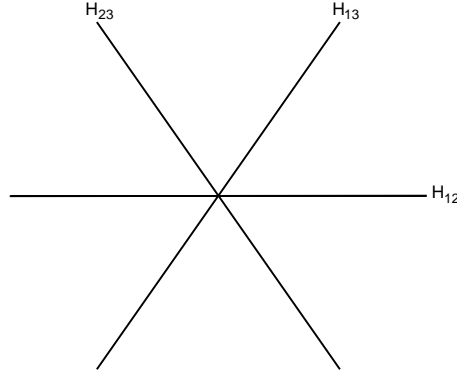We will become more familiar with this particular arrangement as we continue.

Figure 3.4: The braid arrangement, $\mathcal{A}_2$.

Note that $\mathcal{A}_{\ell-1}$ has $\ell!$ chambers and that $\mathcal{A}_{\ell-1}$ consists of $\binom{\ell}{2}$ hyperplanes. This implies that if W is the finite reflection group associated with $\mathcal{A}_{\ell-1}$, then the order of W is $\ell!$. In fact, we have the following result, which we will state without proof.

Proposition 3.14. *If W is the finite reflection group associated to $\mathcal{A}_{\ell-1}$, then W is of type $A_{\ell-1}$. That is, W is isomorphic to the symmetric group on ℓ letters.* $\qquad\square$

If H_{ij} is a hyperplane of $\mathcal{A}_{\ell-1}$, then the reflection $s_{H_{ij}}$ acts on $x \in \mathbb{R}^\ell$ by permuting the ith and j^{th} components of x. Thus, we can think of $s_{H_{ij}}$ as a transposition of the symmetric group. We will make no distinction between the group of reflections of $\mathcal{A}_{\ell-1}$ and S_ℓ.

By Proposition 3.14, we can identify the chambers of $\mathcal{A}_{\ell-1}$ with the permutations of the symmetric group. In fact, if C_0 is the base chamber of $\mathcal{A}_{\ell-1}$ consisting of walls $H_1, \ldots, H_n$, then we can identify each chamber of $\mathcal{A}$ with a product consisting of the transpositions $s_{H_1}, \ldots, s_{H_n}$.

It is clear that the complement of the braid arrangement, $M(\mathcal{A}_{\ell-1})$, consists of all vectors $z = (z_1, \ldots, z_\ell) \in \mathbb{C}^\ell$ such that $z_1 \neq \cdots \neq z_\ell$. In other words, $M(\mathcal{A}_{\ell-1})$ can be thought of as the set of ordered ℓ-tuples of distinct complex numbers. To make this more precise we will introduce the following definition.

Definition 3.15. Define $\hat{F}_\ell(\mathbb{C})$ to be the space of labelled configurations of ℓ distinct points in $\mathbb{C}$.
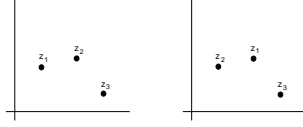
Figure 3.5: Labelled point configurations of $\hat{F}_3(\mathbb{C})$.

Example 3.16. Figure 3.5 shows two examples of labelled configurations of $\hat{F}_3(\mathbb{C})$. In each example the configuration of points is the same, but each is labelled differently, making each a unique element of $\hat{F}_3(\mathbb{C})$.

A point $z = (z_1, \ldots, z_\ell) \in M(\mathcal{A}_{\ell-1})$ determines the labelled configuration of points consisting of $z_1, \ldots, z_\ell$. Also, each labelled configuration of $\hat{F}_\ell(\mathbb{C})$ clearly determines a point $z \in M(\mathcal{A}_{\ell-1})$. Hence, we have the following result.

Proposition 3.17. *The complement of the braid arrangement, $M(\mathcal{A}_{\ell-1})$, is homeomorphic to $\hat{F}_\ell(\mathbb{C})$.* $\qquad\qquad\square$

Chapter 4

Oriented Matroids

The notion of oriented matroid is a natural mathematical concept which presents itself in many different forms. Oriented matroids can thought of as discrete combinatorial abstractions of point configurations over the reals, of convex polytopes, of directed graphs, and of oriented hyperplane arrangements. We will restrict our discussion of oriented matroids to the context of oriented hyperplane arrangements.

A characteristic feature of oriented matroids is the variety of different but equivalent axiom systems. No single axiom system takes priority over the others. We will describe oriented matroids from a covector point of view. But before we describe oriented matroids and how they arise from oriented hyperplane arrangements, we will first introduce some necessary ideas concerning covectors and then list the covector axioms of an oriented matroid.

For our purposes, a *covector* X is simply an element of $\{+,-,0\}^n$. That is, a covector X is an n-tuple where each component X_i is a $+$, $-$, or 0. We will call a covector T a *tope* if $T_i \neq 0$ for all i. The *opposite* of a covector X is $-X$ defined componentwise via

$$
(-X)_i = \begin{cases} - & \text{if } X_i = +, \\ + & \text{if } X_i = -, \\ 0 & \text{if } X_i = 0. \end{cases}
$$

For example, if $X = ++0-$, then $-X = --0+$. The *composition* of two covectors X and Y is $X \circ Y$ defined componentwise via

$$
(X \circ Y)_i = \begin{cases} X_i & \text{if } X_i \neq 0, \\ Y_i & \text{if } X_i = 0. \end{cases}
$$

Note that this composition is associative, but not commutative. The *separation set* of covectors X and Y is $S(X,Y) = \{i : X_i = (-Y)_i \neq 0\}$. Now, we are ready to list the covector axioms for an oriented matroid.

Definition 4.1. A set $\mathcal{L} \subseteq \{+,-,0\}^n$ is the set of covectors of an oriented matroid iff $\mathcal{L}$ satisfies:

(L0) $0 \in \mathcal{L}$,

(L1) $X \in \mathcal{L}$ implies $-X \in \mathcal{L}$,

(L2) $X, Y \in \mathcal{L}$ implies $X \circ Y \in \mathcal{L}$,

(L3) if $X, Y \in \mathcal{L}$ and $i \in S(X, Y)$, then there exists $Z \in \mathcal{L}$ such that $Z_i = 0$ and $Z_j = (X \circ Y)_j = (Y \circ X)_j$ for all $j \notin S(X, Y)$.

We will often refer to a set of covectors $\mathcal{L}$ for an oriented matroid as the oriented matroid itself.

The oriented matroid of a hyperplane arrangement arises in a very natural way. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^\ell$ consisting of hyperplanes $\{H_1, \ldots, H_n\}$. To each face F of $\mathcal{A}$ corresponds a covector X. In this case, we will say that X *labels* F. The oriented matroid associated with $\mathcal{A}$ consists of exactly these covectors. We describe this correspondence below.

Recall that each hyperplane H_i is given by a linear function α_i, so that

$$H_i = \{x \in \mathbb{R}^\ell : \alpha_i(x) = 0\}$$

and

$$H_i{}^+ = \{x \in \mathbb{R}^\ell : \alpha_i(x) > 0\}.$$

The linear functions $\alpha_1, \ldots, \alpha_n$ determine the covectors corresponding to the faces of $\mathcal{A}$. In fact, α_i determines the ith component of each covector. Let F be a face of $\mathcal{A}$ and let $x \in F$. Then the covector X corresponding to F is given by

$$X_i = \begin{cases} + & \text{if } \alpha_i(x) > 0, \\ - & \text{if } \alpha_i(x) < 0, \\ 0 & \text{if } \alpha_i(x) = 0. \end{cases}$$

In the above definition, X_i is independent of the point $x \in F$. However, X_i is dependent upon the numbering of the hyperplanes. A different numbering of the hyperplanes will give rise to a different set of covectors. Yet the set of corresponding covectors will be in one-to-one correspondence with the faces of $\mathcal{A}$, regardless of the numbering. Let $\mathcal{L}(\mathcal{A})$ denote the set of covectors for the oriented hyperplane arrangement $\mathcal{A}$. Likewise, let $\mathcal{T}(\mathcal{A})$ denote the set of topes of $\mathcal{A}$. Note that the topes of $\mathcal{A}$ label exactly the chambers of $\mathcal{A}$, since each α_i is nonzero on a chamber.

There is a very convenient geometric interpretation of the composition of covectors for hyperplane arrangements. Let $\mathcal{L}(\mathcal{A})$ be the set of covectors for an oriented hyperplane arrangement $\mathcal{A}$. Let F and F' be faces of $\mathcal{A}$. Assume that the covectors X and X' label F and F', respectively. Then we can interpret the covector composition of $X \circ X'$ geometrically as "stand on F and take one step towards F'." That is, pick an arbitrary point $x \in F$, and then move a small distance ϵ from x towards F'. The face that you end up in is labelled by $X \circ X'$.

Theorem 4.2. *If $\mathcal{A}$ is an oriented essential hyperplane arrangement, then $\mathcal{L}(\mathcal{A})$ is the set of covectors for an oriented matroid. That is, $\mathcal{L}(\mathcal{A})$ is the oriented matroid for $\mathcal{A}$.*

Proof. Let $\mathcal{L}(\mathcal{A})$ be the set of covectors for an oriented hyperplane arrangement $\mathcal{A}$. We must show that $\mathcal{L}(\mathcal{A})$ satisfies (L0)-(L3). Since $\{0\}$ is a face of $\mathcal{A}$, (L0) is clearly satisfied. To see that (L1) is satisfied, assume that X is in $\mathcal{L}(\mathcal{A})$ and labels the face F of $\mathcal{A}$. Then $-X$ labels the face of $\mathcal{A}$ opposite F. So, $-X$ is in $\mathcal{L}(\mathcal{A})$ and hence (L1) is satisfied. Also, (L2) is clearly satisfied, based upon our discussion of the geometric interpretation of covector composition. Now, let $X, Y \in \mathcal{L}(\mathcal{A})$ and let $i \in S(X, Y)$. Assume that X and Y label the faces F and F', respectively. Geometrically, $i \in S(X, Y)$ implies that the hyperplane H_i separates the faces F and F'. By separates, we mean that for any line segment L joining any two arbitrary points $x(F)$ and $x(F')$ in F and F', respectively, L passes through H_i. Fix such a line segment L. Let G be the face of H_i that L passes through and let Z be the covector that labels G. Since Z lies on H_i, $Z_i = 0$. Now, let $j \notin S(X, Y)$. Then H_j does not separate X and Y. Clearly, the faces labelled by $X \circ Y$, $Y \circ X$, and Z are all on the same side of H_j. Thus,

$$Z_j = (X \circ Y)_j = (Y \circ X)_j.$$

So, (L3) is also satisfied. $\qquad\square$

Example 4.3. Let $\mathcal{A}_2$ be oriented as in Figure 4.1. Given the orientation on $\mathcal{A}_2$, the faces of $\mathcal{A}_2$ are labelled by their corresponding covectors.
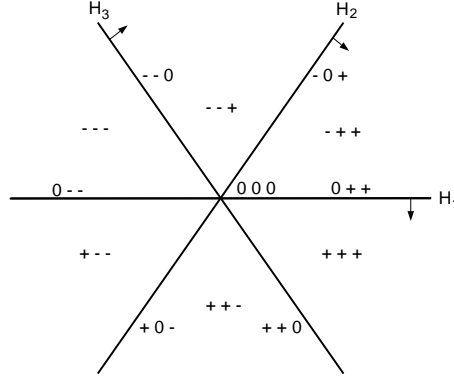

Figure 4.1: Covectors of $\mathcal{A}_2$

We define the *rank* of an oriented matroid $\mathcal{L}$ to be the length of the maximal chain in the poset of covectors. If $\mathcal{L}(\mathcal{A})$ is the oriented matroid of a hyperplane arrangement $\mathcal{A}$, then the rank of $\mathcal{L}(\mathcal{A})$ is the codimension of $\bigcap_{H \in \mathcal{A}} H$.

If $\mathcal{L}$ is an oriented matroid, then we say that $\mathcal{L}$ is *loop-free* if for every i, there exists at least one covector where the ith component is nonzero. Clearly, if $\mathcal{A}$ is a hyperplane arrangement, then $\mathcal{L}(\mathcal{A})$ is loop-free, since there is at least one "nonzero" face on either side of each hyperplane in $\mathcal{A}$.

Now, define the partial order "$\leq$" on the set $\{+, -, 0\}$ via $0 < +$, $0 < -$, with $+$ and $-$ incomparable. See Figure 4.2. This induces a partial order on covectors in which covectors are compared componentwise. That is, for covectors X and X'

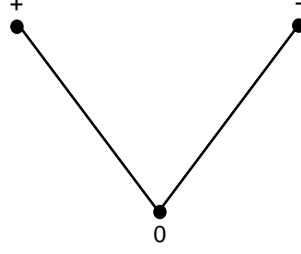$$X \leq X' \text{ iff } X_i \leq X'_i,$$

Figure 4.2: The partial ordering on $\{+, -, 0\}$

for all i. It is easy to check that this ordering on covectors is consistent with the ordering on faces. That is, if X and X' are the covectors corresponding to F and F', respectively,

$$X \le X' \text{ iff } F \le F'.$$

Thus, $\mathcal{F}(\mathcal{A})$ and $\mathcal{L}(\mathcal{A})$ are isomorphic as posets.

Example 4.4. Let $\mathcal{A}_2$ be oriented as in Example 4.3. Then the covectors of $\mathcal{A}_2$, given this orientation, determine the poset for $\mathcal{L}(\mathcal{A}_2)$ in Figure 4.3.
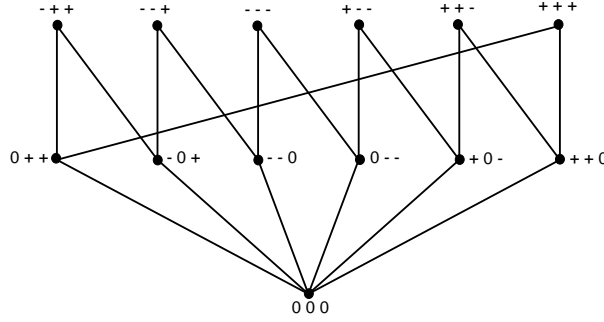
Figure 4.3: Poset of the oriented matroid for $\mathcal{A}_2$

There is an almost automatic identification between the covectors of the oriented matroid of a hyperplane arrangement $\mathcal{A}$ and the faces of $\mathcal{A}$. Since $\mathcal{F}(\mathcal{A})$ and $\mathcal{L}(\mathcal{A})$ are isomorphic as posets, it will be very convenient for us to use F to refer to both the face of $\mathcal{A}$ and the covector which labels F. Similarly, we will use C to denote both a chamber of $\mathcal{A}$ as well as the tope that labels C. We will stick to this convention for the remainder of the paper.

Arrangements of Pseudospheres   Not every oriented matroid $\mathcal{L}$ arises from a hyperplane arrangement $\mathcal{A}$. However, every oriented matroid can be realized as an arrangement of oriented pseudospheres. A *pseudosphere* S is a locally flat PL-submanifold of $S^{\ell-1}$ which is homeomorphic to $S^{\ell-2}$. S divides $S^{\ell-1}$ into two components, S^+ and S^-, called *pseudohemispheres*, each of which is homeomorphic to $B^{\ell-1}$.

25

We define an *arrangement of pseudospheres* $\mathcal{S}$ to be a finite set of pseudospheres $\{S_1, \ldots, S_n\}$ in $S^{\ell-1}$ such that

(S1) Every non-empty intersection $S_J = \bigcap_{i \in J} S_i$ is homeomorphic to a sphere of some dimension for every $J \subseteq \{1, \ldots, n\}$.

(S2) For every non-empty intersection S_J and every $i \notin \{1, \ldots, n\}$, the intersection $S_J \cap S_i$ is a pseudosphere in S_J with sides $S_J \cap S_i^+$ and $S_J \cap S_i^-$.

If $S_{\{1,\ldots,n\}} = \emptyset$, then $\mathcal{S}$ is called *essential*. If a positive side and negative side of each pseudosphere has been chosen, then we say that $\mathcal{S}$ is an *oriented arrangement of pseudospheres*. That is, if S_i^+ and S_i^- have been specified for each pseudosphere S_i, then $\mathcal{S}$ has been oriented. By saying that an arrangement of pseudospheres $\mathcal{S}$ is *centrally symmetric* we mean that each pseudosphere S_i of $\mathcal{S}$ is invariant under the antipodal mapping.

We form the oriented matroid for an arrangement of pseudospheres $\mathcal{L}(\mathcal{S})$ in much the same way that we formed the oriented matroid for an arrangement of hyperplanes. The pseudospheres subdivide $S^{\ell-1}$ into cells. Let $\mathcal{F}(\mathcal{S})$ denote the set of cells of $\mathcal{S}$. Each $F \in \mathcal{F}(\mathcal{S})$ can be labeled by a covector determined by the pseudohemispheres that F lies in. We are now prepared to understand the statement of the following theorem, called the Topological Representation Theorem, which is concerned with the realizability of an oriented matroid. For a proof, see [2].

Theorem 4.5. *Let $\mathcal{L} \subseteq \{+, -, 0\}^n$. Then $\mathcal{L}$ is the set of covectors of a loop-free oriented matroid of rank ℓ iff $\mathcal{L}$ is isomorphic to the face poset $\mathcal{F}(\mathcal{S})$ for some oriented arrangement of pseudospheres $\mathcal{S}$ in $S^{\ell-1}$, which is essential and centrally symmetric.* $\square$

Note that if $\mathcal{A}$ is a hyperplane arrangement, then $\mathcal{F}(\mathcal{A})$ is independent of both the numbering and the orientation of the hyperplanes of $\mathcal{A}$. However, if we renumber or reorient the hyperplanes of $\mathcal{A}$, then we will end up with a different set of covectors for $\mathcal{L}(\mathcal{A})$. But sets of covectors which arise from renumbering or reorienting the hyperplanes of $\mathcal{A}$ are isomorphic as posets. This is clear from the Topological Representation Theorem.

Chapter 5

The Salvetti Complex

Let $\mathcal{A}$ be an essential arrangement of hyperplanes in $\mathbb{R}^\ell$. Throughout the remainder of this paper, we will assume that $S^{\ell-1}$ and B^ℓ are provided with the simplicial decomposition described in Chapter 2. The following definition was introduced by W. Arvola in [1].

Definition 5.1. We define the abstract *Salvetti complex* of a hyperplane arrangement $\mathcal{A}$, denoted $\mathrm{Sal}(\mathcal{A})$, via

$$\mathrm{Sal}(\mathcal{A}) = \{(F, C) : F \in \mathcal{L}(\mathcal{A}), C \in \mathcal{T}(\mathcal{A}), F \leq C\}.$$

Recall that we use the symbol F not only to denote a covector, but also the face of the arrangement that F labels. Similarly, we use C to denote both a tope and the chamber that C labels. So, the pair (F, C) represents a pair of covectors and a pair of faces. We will only make a distiction when the context is unclear.

Example 5.2. Let $\mathcal{A}$ be the oriented hyperplane arrangement given in Figure 5.1. The faces of the arrangement are labelled with their corresponding covector, given the numbering and orientation of the hyperplanes. Then $\mathrm{Sal}(\mathcal{A})$ consists of the elements
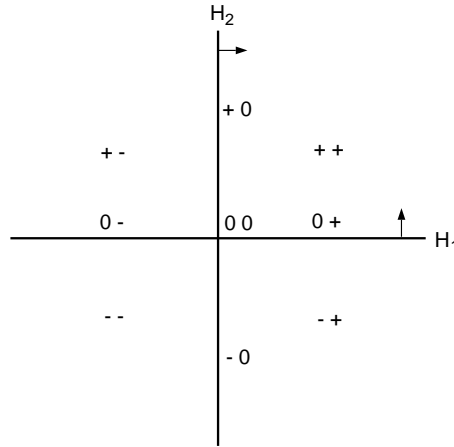

Figure 5.1: Covectors of a hyperplane arrangement

27

$$
\begin{array}{llll}
(++,++) & (-+,-+) & (--,--) & (+-,+-) \\
(0+,++) & (0+,-+) & (-0,--) & (0-,+-) \\
(+0,++) & (-0,-+) & (0-,--) & (+0,+-) \\
(00,++) & (00,-+) & (00,--) & (00,+-)
\end{array}
$$

If $\mathcal{A}$ is a real hyperplane arrangement, then we define a partial order on $\mathrm{Sal}(\mathcal{A})$ via

$$(F', C') \leq (F, C) \text{ iff } F \leq F' \text{ and } F' \circ C = C'.$$

Recall that covector composition is defined componentwise via

$$
(F' \circ C)_j = \begin{cases} (F')_j & \text{if } (F')_j \neq 0, \\ (C)_j & \text{if } (F')_j = 0. \end{cases}
$$

For the remainder of this paper, we assume that $\mathrm{Sal}(\mathcal{A})$ is provided with the partial ordering described above. That is, $\mathrm{Sal}(\mathcal{A})$ is a poset consisting of pairs (F, C), where $F \in \mathcal{F}(\mathcal{A})$, $C \in \mathcal{C}(\mathcal{A})$, and $F \leq C$. We may sometimes refer to a pair (F, C) of $\mathrm{Sal}(\mathcal{A})$ as a face.

Example 5.3. Let $\mathrm{Sal}(\mathcal{A})$ consist of those elements listed in Example 5.2. The poset for $\mathrm{Sal}(\mathcal{A})$ is given in Figure 5.2. Given this poset, we can realize $\mathrm{Sal}(\mathcal{A})$ as the torus of Figure 5.3.
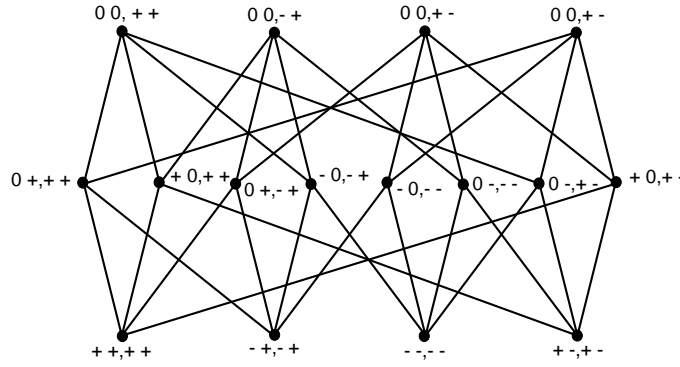
Figure 5.2: Poset for $\mathrm{Sal}(\mathcal{A})$

The face poset for $\mathrm{Sal}(\mathcal{A})$ determines an abstract simplicial complex, by taking the barycentric subdivision. A chain

$$\omega_0 < \cdots < \omega_n$$

of faces of $\mathrm{Sal}(\mathcal{A})$ determines an n-simplex

$$\Omega = (\omega_0 < \cdots < \omega_n).$$

We will use $\mathrm{Sal}'(\mathcal{A})$ to denote the abstract simplicial complex which results from this construction. We will think of each element $\omega \in \mathrm{Sal}(\mathcal{A})$ as a vertex of the simplicial complex, $\mathrm{Sal}'(\mathcal{A})$. Also, every Ω of $\mathrm{Sal}'(\mathcal{A})$ represents both a chain of faces of $\mathrm{Sal}(\mathcal{A})$ and a simplex.

If

$$\Omega = (\omega_0 < \cdots < \omega_n)$$

Figure 5.3: A realization of $\mathrm{Sal}(\mathcal{A})$

and

$$\Omega' = (\omega'_0 < \cdots < \omega'_r)$$

are simplices of $\mathrm{Sal}'(\mathcal{A})$, then $\Omega \leq \Omega'$ iff

$$\omega_0 < \cdots < \omega_n$$

is a subchain of

$$\omega'_0 < \cdots < \omega'_r.$$

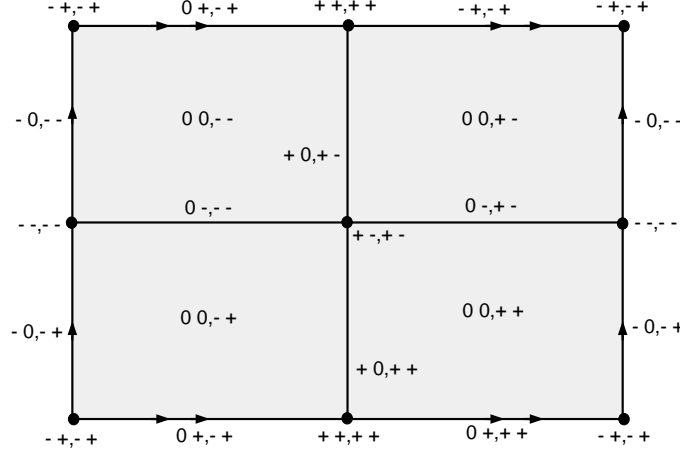We can realize the abstract simplicial complex $\mathrm{Sal}'(\mathcal{A})$ in $\mathbb{C}^\ell$ by forming the *concrete simplicial Salvetti complex*, denoted $|\mathrm{Sal}'(\mathcal{A})|$. Recall that $\mathcal{A}$ determines a simplicial decomposition of B^ℓ. Now, for every pair (F, C) in $\mathrm{Sal}(\mathcal{A})$ we set

$$z(F, C) = x(F) + ix(C).$$

Note that since C is a chamber of $\mathcal{A}$, $x(C) \notin H$ for each hyperplane H of $\mathcal{A}$. It follows that $z(F, C) \notin H + iH$ for any H. Thus, $z(F, C) \in M(\mathcal{A})$.

Lemma 5.4. *Let* $(F, C), (F', C') \in \mathrm{Sal}(\mathcal{A})$ *with* $(F, C) < (F', C')$. *Then* $F' < F$.

Proof. Assume that $F = F'$. Then $F \circ C' = F' \circ C' = C'$ since $F' \leq C'$. But $F \circ C' = C$, since $(F, C) < (F', C')$. So, $C = C'$. This implies that $(F, C) = (F', C')$, which is a contradiction. Therefore, $F' < F$. $\square$

Lemma 5.5. *If* $\Omega = (\omega_0 < \cdots < \omega_n)$ *is a simplex of* $\mathrm{Sal}'(\mathcal{A})$, *then the points* $z(\omega_0), \ldots, z(\omega_n)$ *are geometrically independent.*

Proof. Let $(\omega_k < \cdots < \omega_n)$ be a simplex of $\mathrm{Sal}'(\mathcal{A})$. Set $\omega_j = (F_j, C_j)$. Then each $z(\omega_j) = x(F_j) + ix(C_j)$. Assume that $x(F_{k+1}), \ldots, x(F_n)$ are geometrically independent. Then $z(\omega_{k+1}), \ldots, z(\omega_n)$ are geometrically independent. Also, $x(F_{k+1}), \ldots, x(F_n)$ lie in the support of F_{k+1}, since $F_n \leq \cdots \leq F_{k+1}$. By Lemma 5.4, $F_{k+1} < F_k$. This implies that $x(F_k)$ does not lie in the support of

29

F_{k+1}. So, $x(F_k), \ldots, x(F_n)$ are geometrically independent. This implies that $z(\omega_k), \ldots, z(\omega_n)$ are geometrically independent. This induction shows that $z(\omega_0), \ldots, z(\omega_n)$ are geometrically independent. $\qquad\square$

The concrete Salvetti complex $|\operatorname{Sal}'(\mathcal{A})|$ is a simplicial complex having $\{z(\omega) : \omega \in \operatorname{Sal}(\mathcal{A})\}$ as its set of vertices. The vertices $z(\omega_0), \ldots, z(\omega_n)$ of $|\operatorname{Sal}'(\mathcal{A})|$ form an n-simplex of $|\operatorname{Sal}'(\mathcal{A})|$ exactly when $\omega_0, \cdots, \omega_n$ are the elements of a chain Ω. If

$$\Omega = (\omega_0 < \cdots < \omega_n)$$

is a simplex of $\operatorname{Sal}'(\mathcal{A})$, then we will use $z(\Omega)$ to denote the simplex

$$z(\omega_0) \vee \cdots \vee z(\omega_n)$$

of $|\operatorname{Sal}'(\mathcal{A})|$.

Recall from Chapter 2, that we define the cone of a cell ϕ of $S^{\ell-1}$ to be

$$K(\phi) = \{\lambda x : x \in \phi \text{ and } \lambda > 0\}.$$

For ease of notation, we will denote the cone of the star of a vertex v of $S^{\ell-1}$ by $\hat{K}(v)$. That is, $\hat{K}(v) = K(Star(v))$.

Example 5.6. The shaded region of Figure 5.4 shows the cone of the star of the vertex v_0 on S^1.
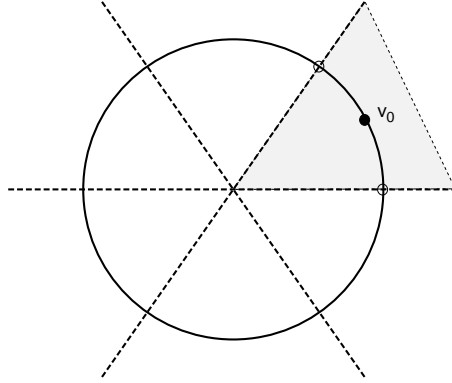

Figure 5.4: The cone of the star of a vertex

Now, we will list a series of lemmas which will be used to prove that $|\operatorname{Sal}'(\mathcal{A})|$ has the same homotopy type as the complement $M(\mathcal{A})$.

Lemma 5.7. Let $(F, C) \in \operatorname{Sal}(\mathcal{A})$ for some essential hyperplane arrangement $\mathcal{A}$. Then

$$F + iC_F \subseteq M(\mathcal{A}).$$

30

Proof. Let $z = x + iy \in F + iC_F$. Then $x \in F$ and $y \in C_F$. Since C_F is the unique chamber of $\mathcal{A}_F$ which contains C, y is not contained in any of the hyperplanes that F may be contained in. Hence $x + iy \notin H + iH$ for every hyperplane H of $\mathcal{A}$. So, $z \in M(\mathcal{A})$. Therefore,

$$F + iC_F \subseteq M(\mathcal{A}).$$

$\square$

Lemma 5.8. *Let*

$$\phi = x(F_0) \vee \cdots \vee x(F_n)$$

be a simplex of $S^{\ell-1}$ with

$$\{0\} \neq F_0 \leq \cdots \leq F_n.$$

Then $K(\phi) \subseteq \hat{K}(x(F_i))$ for all i.

Proof. Since each $x(F_i)$ is a vertex of ϕ, $\phi \subseteq \text{Star}(x(F_i))$ for each i. Hence, $K(\phi) \subseteq \hat{K}(x(F_i))$ for each i. $\square$

Lemma 5.9. *Let $F \in \mathcal{F}(\mathcal{A})$ and let D be a chamber of $\mathcal{A}_F$. Then there exists a unique chamber C of $\mathcal{A}$ such that $C_F = D$ and $F \leq C$.*

Proof. Let $F \in \mathcal{F}(\mathcal{A})$ and let D be a chamber of $\mathcal{A}_F$. Clearly, there exists at least one chamber C_0 of $\mathcal{A}$ such that $(C_0)_F = D$. Since D is a chamber of $\mathcal{A}_F$, D is a partial covector having one entry for each hyperplane H_j which contains the face F. That is, D_j exists only when $F_j = 0$. Since $(C_0)_F = D$, $(C_0)_j = D_j$ when $F_j = 0$. Furthermore, if C is any chamber with $C_F = D$, then $C_j = D_j$ when $F_j = 0$. Also, if $F \leq C$, then $C_j = F_j$ whenever $F_j \neq 0$. Thus, there can be at most one chamber C satisfying $C_F = D$ and $f \leq C$ and such a chamber must be given by

$$C_j = \begin{cases} D_j & \text{if } H_j \in \mathcal{A}_F, \\ F_j & \text{if } F_j \notin \mathcal{A}_F. \end{cases}$$

On the other hand, the sign vector given by this formula is precisely $F \circ C_0$, which is in $\mathcal{L}(\mathcal{A})$ by Theorem 4.2 and (L2). Clearly, C is a tope. $\square$

Lemma 5.10. *Let $(F_0, C_0), \ldots, (F_n, C_n)$ be elements of $\text{Sal}(\mathcal{A})$, where*

$$(F_0, C_0) \leq \cdots \leq (F_n, C_n).$$

Then $C_n \subseteq (C_i)_{F_i}$ for all i.

Proof. Recall that $(F_i, C_i) \leq (F_{i+1}, C_{i+1})$ implies that

$$F_{i+1} \leq F_i \text{ and } F_i \circ C_{i+1} = C_i.$$

We will prove that $C_n \subseteq (C_i)_{F_i}$ for all i by induction. First, $C_n \subseteq (C_n)_{F_n}$, by definition. Now, assume that $C_n \subseteq (C_{i+1})_{F_{i+1}}$. Fix j with $(F_i)_j = 0$. Then $(F_{i+1})_j = 0$, since $F_{i+1} \leq F_i$. Now, since $(F_i, C_i) \leq (F_{i+1}, C_{i+1})$,

$$F_i \circ C_{i+1} = C_i.$$

But $(F_i \circ C_{i+1})_j = (C_{i+1})_j$, by definition, since $(F_i)_j = 0$. So, $(C_i)_j = (C_{i+1})_j$. Also, $(C_n)_j = (C_{i+1})_j$ since $C_n \subseteq (C_{i+1})_{F_{i+1}}$. Hence,

$$(C_i)_j = (C_{i+1})_j = (C_n)_j,$$

whenever H_j is a hyperplane of $\mathcal{A}_{F_i}$. Therefore, $C_n \subseteq (C_i)_{F_i}$. $\square$

Now, we are ready to state the main result of this paper, the proof of which will rely on the above lemmas and the Nerve Theorem. The proof that we give follows the same framework as a proof given by L. Paris in [9]. In [9], Paris constructs a concrete Salvetti complex from pairs (F, C), where F is a face of an essential arrangement $\mathcal{A}$ and C is an arbitrary chamber. He does not require that F be a face of C, as we have. From each pair (F, C), Paris forms the vertex

$$z(F, C) = x(F) + ix(C_F)$$

in $\mathbb{C}^\ell$. He points out that $z(F_1, C_1) = z(F_2, C_2)$ iff $F_1 = F_2$ and $(C_1)_{F_1} = (C_2)_{F_2}$. Our Lemma 5.9 shows that we get the same vertex set as Paris.

Theorem 5.11 (Salvetti [11], Paris [9]). *Let $\mathcal{A}$ be an oriented hyperplane arrangement. Then $|\mathrm{Sal}'(\mathcal{A})|$ has the same homotopy type as $M(\mathcal{A})$.*

Proof. First, assume that B^ℓ is provided with the simplicial decompostion determined by $\mathcal{A}$. Now, for every $\omega \in \mathrm{Sal}(\mathcal{A})$, we will associate an open convex subset $U(\omega)$ of $M(\mathcal{A})$. Let (F, C) be an element of $\mathrm{Sal}(\mathcal{A})$. If $F = \{0\}$, then we set

$$U(F, C) = \mathbb{R}^\ell + iC.$$

If $F \neq \{0\}$, then we set

$$U(F, C) = \hat{K}(x(F)) + iC_F.$$

By Lemma 5.9, each the $U(\omega)$ are distinct. Also, $U(\omega)$ is an open convex set for each $\omega \in \mathrm{Sal}(\mathcal{A})$. Then any nonempty intersection of elements of

$$\mathcal{U} = \{U(\omega) : \omega \in \mathrm{Sal}(\mathcal{A})\}$$

will be convex and thus contractible. It is also clear that $z(\omega) \in U(\omega)$ for every ω of $\mathrm{Sal}(\mathcal{A})$.

Now, we will prove the following four assertions successively.

(1) $U(\omega) \subseteq M(\mathcal{A})$ for every $\omega \in \mathrm{Sal}(\mathcal{A})$.

(2) $M(\mathcal{A}) \subseteq \bigcup_{\omega \in \mathrm{Sal}(\mathcal{A})} U(\omega)$.

(3) Let $z(\omega_0), \ldots, z(\omega_n)$ be $(n+1)$ distinct vertices of $|\mathrm{Sal}'(\mathcal{A})|$. If

$$U(\omega_0) \cap \cdots \cap U(\omega_n) \neq \emptyset,$$

then $z(\omega_0), \ldots, z(\omega_n)$ are the vertices of a simplex $z(\Omega)$ of $|\mathrm{Sal}'(\mathcal{A})|$.

(4) If $z(\omega_0), \ldots, z(\omega_n)$ are the vertices of a simplex $z(\Omega)$ of $|\mathrm{Sal}'(\mathcal{A})|$, then

$$U(\omega_0) \cap \cdots \cap U(\omega_n) \neq \emptyset.$$

Clearly, (1) and (2) prove that $\mathcal{U}$ is an open covering of $M(\mathcal{A})$. Also, assertions (3) and (4) prove that $\mathrm{Sal}'(\mathcal{A})$ is the nerve of $\mathcal{U}$. This implies, by the Nerve Theorem (Theorem 1.18), that $|\mathrm{Sal}'(\mathcal{A})|$ has the same homotopy type as $M(\mathcal{A})$.

(1) Let $(F, C) \in \mathrm{Sal}(\mathcal{A})$. If $F = \{0\}$, then

$$U(F, C) = \mathbb{R}^\ell + iC.$$

Since $z = x + iy \in H + iH$ iff $x \in H$ and $y \in H$, $U(F, C)$ is clearly contained in $M(\mathcal{A})$. Now, assume that $F \neq \{0\}$. Then

$$U(F, C) = \hat{K}(x(F)) + iC_F.$$

Pick $z = x + iy \in U(F, C)$. Then $x \in \hat{K}(x(F))$. Thus, there exists a simplex ϕ of $S^{\ell-1}$ such that $x(F)$ is a vertex of ϕ and $x \in K(\phi)$. Say

$$\phi = x(F_0) \vee \cdots \vee x(F_n),$$

with

$$\{0\} \neq F_0 < \cdots < F_n,$$

where $F = F_i$ for some i. By Lemma 2.6, $K(\phi) \subseteq F_n$, which implies that $x \in F_n$. Also, $F \leq F_n$. So, $C_F \subseteq C_{F_n}$, which implies that $y \in C_{F_n}$. Therefore,

$$z = x + iy \in F_n + iC_{F_n}.$$

By Lemma 5.7, $F_n + iC_{F_n} \subseteq M(\mathcal{A})$ and so $z = x + iy \in M(\mathcal{A})$. Hence, $U(\omega) \subseteq M(\mathcal{A})$ for every $\omega \in \mathrm{Sal}(\mathcal{A})$.

(2) Let $z = x + iy \in M(\mathcal{A})$. If $x = 0$, then $x \in H$ for every $H \in \mathcal{A}$. Thus, $y \notin H$ for every $H \in \mathcal{A}$, since $x + iy \in M(\mathcal{A})$. So, there must exist a chamber C of $\mathcal{A}$ such that $y \in C$. Therefore,

$$z = x + iy \in \mathbb{R}^\ell + iC.$$

But $(\mathbb{R}^\ell + iC) = U(\{0\}, C)$. Hence,

$$z = x + iy \in U(\{0\}, C).$$

Now, assume that $x \neq 0$. Then there exists a simplex ϕ of $S^{\ell-1}$ such that $x \in K(\phi)$. Say

$$\phi = x(F_0) \vee \cdots \vee x(F_n)$$

where

$$\{0\} \neq F_0 < \cdots < F_n.$$

Since $x \in K(\phi) \subseteq F_n$ and $z = x + iy \in M(\mathcal{A})$, there is no hyperplane $H \in \mathcal{A}$ containing F_n which also contains y. Hence, there exists a chamber D of $\mathcal{A}_{F_n}$ such that $y \in D$. By Lemma 5.9, there exists a chamber C of $\mathcal{A}$ such that $C_{F_n} = D$ and $F_n \leq C$, so that $(F_n, C) \in \mathrm{Sal}(\mathcal{A})$. Then

$$z = x + iy \in K(\phi) + iC_{F_n}.$$

Also, by Lemma 5.8, $K(\phi) \subseteq \hat{K}(x(F_n))$. So,

$$K(\phi) + iC_{F_n} \subseteq \hat{K}(x(F_n)) + iC_{F_n} = U(F_n, C).$$

Therefore,

$$z = x + iy \in U(F_n, C).$$

Thus, $M(\mathcal{A}) \subseteq \bigcup_{\omega \in \mathrm{Sal}(\mathcal{A})} U(\omega)$.

(3) Let $z(\omega_0), \ldots, z(\omega_n)$ be $(n+1)$ distinct vertices of $|\mathrm{Sal}'(\mathcal{A})|$ such that

$$U(\omega_0) \cap \cdots \cap U(\omega_n) \neq \emptyset.$$

Then $\omega_0, \ldots, \omega_n$ are $(n+1)$ distinct elements of $\mathrm{Sal}(\mathcal{A})$. Set

$$\omega_i = (F_i, C_i)$$

for all i and let

$$z = x + iy \in \bigcap_{i=0}^{n} U(F_i, C_i).$$

We have two cases to consider: $F_i = \{0\}$ for some i and $F_i \neq \{0\}$ for all i.

Case A: Without loss of generality, assume that $F_0 = \{0\}$. First, suppose that there exists an j such that $F_j = \{0\}$. Then

$$z = x + iy \in U(\omega_0) \cap U(\omega_j) = (\mathbb{R}^\ell + iC_0) \cap (\mathbb{R}^\ell + iC_j).$$

But then $y \in C_0 \cap C_j$. So, $C_0 \cap C_j \neq \emptyset$, which implies that $C_0 = C_j$. Then

$$\omega_0 = (\{0\}, C_0) = (\{0\}, C_j) = \omega_j.$$

But this contradicts the fact that $\omega_0, \ldots, \omega_n$ are $(n+1)$ distinct elements of $\mathrm{Sal}(\mathcal{A})$. Therefore, $F_j \neq \{0\}$ for all $j > 0$. Now, since $x \in \hat{K}(x(F_i))$ for each i, there exists a simplex ϕ_i of $S^{\ell-1}$ such that $x(F_i)$ is a vertex of ϕ_i and $x \in K(\phi_i)$, for all i. This implies that

$$\phi_1 = \cdots = \phi_n,$$

since

$$\{K(\phi) : \phi \text{ a simplex of } S^{\ell-1}\}$$

is a partition of $\mathbb{R}^\ell - \{0\}$. Hence, $x(F_1), \ldots, x(F_n)$ are the vertices of some simplex ϕ of $S^{\ell-1}$.

Without loss of generality, assume that

$$\{0\} = F_0 < F_1 \leq \cdots \leq F_n.$$

Now, we need to show that $F_{i+1} \circ C_i = C_{i+1}$, for all i. We will show that

$$(F_{i+1} \circ C_i)_j = (C_{i+1})_j$$

for each j. Recall that

$$(F_{i+1} \circ C_i)_j = \begin{cases} (F_{i+1})_j & \text{if } (F_{i+1})_j \neq 0, \\ (C_i)_j & \text{if } (F_{i+1})_j = 0. \end{cases}$$

Fix j. First, assume that $(F_{i+1})_j \neq \{0\}$. Then

$$(F_{i+1} \circ C_i)_j = (F_{i+1})_j.$$

34

But since $F_{i+1} \leq C_{i+1}$, $(F_{i+1})_j = (C_{i+1})_j$. Hence,

$$(F_{i+1} \circ C_i)_j = (C_{i+1})_j.$$

Now, assume that $(F_{i+1})_j = \{0\}$. Then F_{i+1} lies inside the hyperplane H_j. Also,

$$(F_{i+1} \circ C_i)_j = (C_i)_j$$

by definition. We need to show that $(C_i)_j = (C_{i+1})_j$. We know that

$$(C_i)_{F_i} \cap (C_{i+1})_{F_{i+1}} \neq \emptyset$$

since both factors contain y, and that $F_i \leq F_{i+1}$. Then $\mathcal{A}_{F_{i+1}} \subseteq \mathcal{A}_{F_i}$, which implies that $(C_i)_{F_i} \subseteq (C_i)_{F_{i+1}}$. Since

$$(C_i)_{F_i} \cap (C_{i+1})_{F_{i+1}} \neq \emptyset$$

and $(C_i)_{F_i} \subseteq (C_i)_{F_{i+1}}$,

$$(C_i)_{F_{i+1}} \cap (C_{i+1})_{F_{i+1}} \neq \emptyset.$$

So,

$$(C_i)_{F_{i+1}} = (C_{i+1})_{F_{i+1}}.$$

Since $H_j \in \mathcal{A}_{F_{i+1}}$, C_i and C_{i+1} lie on the same side of H_j. Hence, $(C_i)_j = (C_{i+1})_j$. Therefore,

$$(F_{i+1} \circ C_i)_j = (C_{i+1})_j.$$

Hence

$$F_{i+1} \circ C_i = C_{i+1}.$$

So, $(F_{i+1}, C_{i+1}) \leq (F_i, C_i)$ for all i. That is,

$$(F_n, C_n) \leq \cdots \leq (F_1, C_1) \leq (F_0, C_0).$$

Case B: Assume that $F_0 \neq \{0\}$. Since $x \in \hat{K}(x(F_i))$ for each i, there exists a simplex ϕ_i of $S^{\ell-1}$ such that $x(F_i)$ is a vertex of ϕ_i and $x \in K(\phi_i)$, for i. This implies that

$$\phi_0 = \phi_1 = \cdots = \phi_n,$$

and so, $x(F_0), \ldots, x(F_n)$ are the vertices of some simplex ϕ of $S^{\ell-1}$, just as in the previous case.

Without loss of generality, assume that

$$\{0\} \neq F_0 < F_1 \leq \cdots \leq F_n.$$

The proof that $F_{i+1} \circ C_i = C_{i+1}$, for each i is identical to the proof above. So, we have that

$$(F_n, C_n) \leq \cdots \leq (F_1, C_1) \leq (F_0, C_0).$$

Therefore, in either case, we have that

$$(F_n, C_n), \ldots, (F_1, C_1), (F_0, C_0)$$

are the vertices of an $(n + 1)$ simplex Ω of $\mathrm{Sal}'(\mathcal{A})$. It follows that

$$z(F_n, C_n), \ldots, z(F_1, C_1), z(F_0, C_0)$$

are the vertices of an $(n + 1)$ simplex of $|\mathrm{Sal}'(\mathcal{A})|$.

(4) Let

$$z(\Omega) = z(\omega_0) \vee \cdots \vee z(\omega_n)$$

be a simplex of $|\mathrm{Sal}'(\mathcal{A})|$, where

$$\omega_0 \leq \cdots \leq \omega_n.$$

Then

$$\Omega = (\omega_0 < \cdots < \omega_n)$$

is a simplex of $\mathrm{Sal}'(\mathcal{A})$. Set

$$\omega_i = (F_i, C_i)$$

for each i. Then

$$F_0 \geq \cdots \geq F_n.$$

Note the change in the inequality above. Let

$$\phi = x(F_0) \vee x(F_1) \vee \cdots \vee x(F_n).$$

Then ϕ is a simplex of B^ℓ. We have two cases to consider: $F_n = \{0\}$ and $F_n \neq \{0\}$.

Case A: Assume that $F_n = \{0\}$. Consider the simplex

$$\phi' = x(F_0) \vee \cdots \vee x(F_{n-1})$$

of $S^{\ell-1}$. Let $x \in K(\phi')$ and $y \in C_n$. Set

$$z = x + iy.$$

Clearly, we have

$$z \in \mathbb{R}^\ell + iC_n = U(F_n, C_n).$$

Since $x \in K(\phi')$, $x \in \hat{K}(x(F_i))$ for each $i = 0, 1, \ldots, n - 1$, by Lemma 5.8. Also, $y \in (C_i)_{F_i}$ for $i = 0, 1, \ldots, n$, by Lemma 5.10. Therefore,

$$z = x + iy \in \hat{K}(x(F_i)) + i(C_i)_{F_i} = U(F_i, C_i)$$

for $i = 0, \ldots, n$. So, it follows that

$$U(\omega_0) \cap \cdots \cap U(\omega_n) \neq \emptyset.$$

Case B: Assume that $F_n \neq \{0\}$. Then ϕ is a simplex of $S^{\ell-1}$. Let $x \in K(\phi)$ and $y \in C_n$. Set

$$z = x + iy.$$

Since $x \in K(\phi)$, $x \in \hat{K}(F_i)$ for every i, by Lemma 5.8. Also, $y \in (C_i)_{F_i}$ for each i, by Lemma 5.10. Therefore,

$$z = x + iy \in \hat{K}(x(F_i)) + i(C_i)_{F_i} = U(F_i, C_i)$$

for each i. It follows that

$$U(\omega_0) \cap \cdots \cap U(\omega_n) \neq \emptyset.$$

$\square$

Corollary 5.12. *Let $\mathcal{A}$ be an essential hyperplane arrangement. Then*

$$|\operatorname{Sal}'(\mathcal{A})| \subseteq M(\mathcal{A}).$$

Proof. The result follows from the fact that each $U(\omega_i) \in \mathcal{U}$ is convex and from steps (1) and (4) of the proof of Theorem 5.11. $\qquad\square$

Let $\mathcal{A}$ be an essential hyperplane arrangement. Given $|\operatorname{Sal}'(\mathcal{A})|$, we can easily form the realization $|\operatorname{Sal}(\mathcal{A})|$ of $\operatorname{Sal}(\mathcal{A})$. For each $\omega \in \operatorname{Sal}(\mathcal{A})$ form the cell

$$|\omega| = \bigcup_{z(\omega) \leq z(\Omega)} z(\Omega),$$

where Ω is a simplex of $\operatorname{Sal}'(\mathcal{A})$. This construction yeilds a regular cell complex which is a sort of anti-order complex. Also, it is clear that $|\operatorname{Sal}'(\mathcal{A})|$ and $|\operatorname{Sal}(\mathcal{A})|$ are homeomorphic as topological spaces. So, by Proposition 1.16, we have the following corollary.

Corollary 5.13. *The regular cell complex $|\operatorname{Sal}(\mathcal{A})|$ has the same homotopy type as $M(\mathcal{A})$.* $\qquad\square$

Recall that in Chapter 3 we proved that the complement of the braid arrangement, $M(\mathcal{A}_{\ell-1})$, is isomorphic to the labelled configuration space $\hat{F}_\ell(\mathbb{C})$. This implies the following result.

Corollary 5.14. *$|\operatorname{Sal}(\mathcal{A}_{\ell-1})|$ has the same homotopy type as $\hat{F}_\ell(\mathbb{C})$.* $\qquad\square$

Complex Covectors There is an alternative description of $\operatorname{Sal}(\mathcal{A})$ which we will now discuss. If $\mathcal{A}$ consists of the hyperplanes $H_1, \ldots, H_n$, then we can form the set of *complex covectors* for the complexified arrangement $\mathcal{A}_{\mathbb{C}}$, where each complex covector is an element of $\{0, +, -, i, -i\}^n$. Let α_j be the defining equation for the complex hyperplane $H_j + iH_j$. For each $z \in \mathbb{C}^\ell$ we associate a vector X_z, which we define componentwise via

$$(X_z)_j = \begin{cases} 0 & \text{if } \alpha_j(z) = 0, \\ + & \text{if } \operatorname{Re}(\alpha_j(z)) > 0, \\ - & \text{if } \operatorname{Re}(\alpha_j(z)) < 0, \\ i & \text{if } \operatorname{Re}(\alpha_j(z)) = 0 \text{ and } \operatorname{Im}(\alpha_j(z)) > 0, \\ -i & \text{if } \operatorname{Re}(\alpha_j(z)) = 0 \text{ and } \operatorname{Im}(\alpha_j(z)) < 0. \end{cases}$$

That is, the defining equation for $H_j + iH_j$ determines the jth entry for each vector X_z. The complex covectors for $\mathcal{A}_{\mathbb{C}}$ consist of all the vectors X_z where $z \in \mathbb{C}^\ell$. Certainly, the set of complex covectors is finite. Note that the complex covectors which label the complement $M(\mathcal{A})$ consist of only nonzero entries.

Now, define the partial order "$\leq$" on the set $\{0, +, -, i, -i\}$ via $0 < +$, $0 < -$, $+ < i$, $- < -i$, $- < i$, and $+ < -i$ with $+$ and $-$ incomparable and i and $-i$ incomparable. See Figure 5.5.

This induces a partial order on complex covectors in which complex covectors are compared componentwise. That is, if X and X' are complex covectors of a complexified arrangement $\mathcal{A}_{\mathbb{C}}$, then

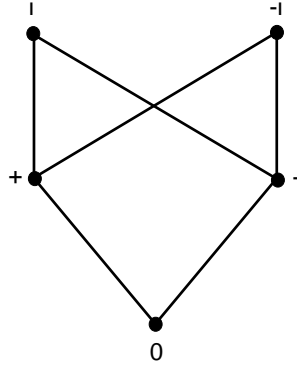$$X \leq X' \text{ iff } X_j \leq X_j'$$

Figure 5.5: The partial ordering on $\{0, +, -, i, -i\}$

for all j.

We can convert each pair $(F, C) \in \operatorname{Sal}(\mathcal{A})$ to a complex covector F^C. Let $\mathcal{L}(\mathcal{A}_\mathbb{C})$ denote the set of nowhere zero complex covectors for a complexified arrangement $\mathcal{A}_\mathbb{C}$. Now, define $f : \operatorname{Sal}(\mathcal{A}) \longrightarrow \mathcal{L}(\mathcal{A}_\mathbb{C})$ via

$$f(F, C) = F^C,$$

where

$$(F^C)_j = \begin{cases} F_j & \text{if } F_j \neq 0, \\ i & \text{if } F_j = 0 \text{ and } C_j = +, \\ -i & \text{if } F_j = 0 \text{ and } C_j = -. \end{cases}$$

Example 5.15. Let $\mathcal{A}$ be the oriented hyperplane arrangment of Figure 5.1. Then the correspondence between (F, C) and F^C for $\mathcal{A}$ is given below.

$$
\begin{array}{rcl}
++, ++ & \longleftrightarrow & +, + \\
0+, ++ & \longleftrightarrow & i, + \\
+0, ++ & \longleftrightarrow & +, i \\
00, ++ & \longleftrightarrow & i, i \\
-+, -+ & \longleftrightarrow & -, + \\
0+, -+ & \longleftrightarrow & -i, + \\
-0, -+ & \longleftrightarrow & -, i \\
00, -+ & \longleftrightarrow & -i, i \\
--, -- & \longleftrightarrow & -, - \\
-0, -- & \longleftrightarrow & -, -i \\
0-, -- & \longleftrightarrow & -i, - \\
00, -- & \longleftrightarrow & -i, -i \\
+-, +- & \longleftrightarrow & +, - \\
0-, +- & \longleftrightarrow & i, - \\
+0, +- & \longleftrightarrow & +, -i \\
00, +- & \longleftrightarrow & i, -i \\
\end{array}
$$

38

Lemma 5.16. *The function f is injective.*

Proof. Let f be defined as above. Assume that $F_1{}^{C_1} = F_2{}^{C_2}$. Then $(F_1{}^{C_1})_j = (F_2{}^{C_2})_j$ for each j. We have three cases to consider.

Case A: Assume that $(F_1{}^{C_1})_j = (F_1)_j$. Then $(F_2{}^{C_2})_j = (F_1)_j$. This forces $(F_2{}^{C_2})_j = (F_2)_j$, since $(F_2{}^{C_2})_j$ is real. Hence $(F_1)_j = (F_2)_j$. Also, since $(F_1, C_1), (F_2, C_2) \in \mathrm{Sal}(\mathcal{A})$, $(F_1)_j \leq (C_1)_j$ and $(F_2)_j \leq (C_2)_j$. Thus, $(F_1)_j = (C_1)_j$ and $(F_1)_j = (C_2)_j$, since $(F_1)_j = (F_2)_j \neq 0$. Hence $(C_1)_j = (C_2)_j$.

Case B: Assume that $(F_1{}^{C_1})_j = i$. Then $(F_2{}^{C_2})_j = i$. Since $(F_1{}^{C_1})_j = i$, $(F_1)_j = 0$ and $(C_1)_j = +$. Also, since $(F_2{}^{C_2})_j = i$, $(F_2)_j = 0$ and $(C_2)_j = +$. Thus, $(F_1)_j = (F_2)_j$ and $(C_1)_j = (C_2)_j$.

Case C: Assume that $(F_1{}^{C_1})_j = -i$. Then $(F_1)_j = (F_2)_j$ and $(C_1)_j = (C_2)_j$, just as in Case B.

So, in any case, $(F_1)_j = (F_2)_j$ and $(C_1)_j = (C_2)_j$. This implies that $(F_1, C_1) = (F_2, C_2)$. Therefore, f is injective. $\qquad\square$

Lemma 5.17. *The function f is surjective.*

Proof. Let $X_z \in \mathcal{L}(\mathcal{A}_{\mathbb{C}})$. Define (F, C) via

$$
(F_j, C_j) = \begin{cases}
(+, +) & \text{if } (X_z)_j = +, \\
(-, -) & \text{if } (X_z)_j = -, \\
(0, +) & \text{if } (X_z)_j = i, \\
(0, -) & \text{if } (X_z)_j = -i.
\end{cases}
$$

Then clearly, $f(F, C) = X_z$. We must show that the covector F labels a face of $\mathcal{A}$, the covector C labels a chamber of $\mathcal{A}$, and that $F \leq C$, so that (F, C) will be an element of $\mathrm{Sal}(\mathcal{A})$. Set $z = x + iy$. From the definition of (F, C), one sees that F labels the face of $\mathcal{A}$ containing x, and C labels the unique chamber having F as a face, such that C_F contains y, which exists by Lemma 5.9. This implies that $(F, C) \in \mathrm{Sal}(\mathcal{A})$. Therefore, f is surjective. $\qquad\square$

The previous two lemmas show that there is a bijection between $\mathrm{Sal}(\mathcal{A})$ and $\mathcal{L}(\mathcal{A}_C)$.
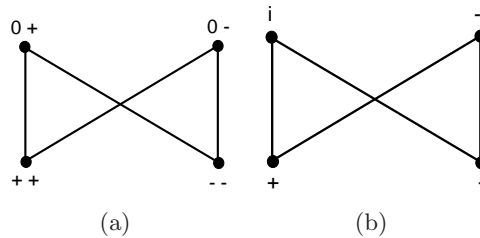

Figure 5.6: Partial ordering on components

Proposition 5.18. $\mathrm{Sal}(\mathcal{A})$ *and* $\mathcal{L}(\mathcal{A}_{\mathbb{C}})$ *are isomorphic as posets.*

Proof. Since there is a bijection between $\mathrm{Sal}(\mathcal{A})$ and $\mathcal{L}(\mathcal{A}_{\mathbb{C}})$, all that we need to show is that the function f preserves order. Let $(F,C) \in \mathrm{Sal}(\mathcal{A})$. Then since $F \leq C$, (F_j, C_j) is one of $(0,+), (0,-), (+,+), (-,-)$. Furthermore, the ordering $(F,C) \leq (F',C')$ defined by $F' \leq F$ and $F \circ C' = C$ can be obtained from a coordinate-wise ordering according to Figure 5.6(a). Since f is defined coordinate-wise via

$$
\begin{aligned}
(0,+) &\longrightarrow \quad i \\
(0,-) &\longrightarrow \quad -i \\
(+,+) &\longrightarrow \quad + \\
(-,-) &\longrightarrow \quad -
\end{aligned}
$$

it is obvious when we compare Figures 5.6(a) and 5.6(b) that f preserves order. Therefore, $\mathrm{Sal}(\mathcal{A})$ and $\mathcal{L}(\mathcal{A}_{\mathbb{C}})$ are isomorphic as posets. $\qquad\square$

The above theorem implies that we may use the set of nowhere zero complex covectors $\mathcal{L}(\mathcal{A}_{\mathbb{C}})$ as an alternate description of $\mathrm{Sal}(\mathcal{A})$.

Chapter 6

Orbit Complexes

Let $\mathcal{A}$ be a reflection arrangement in $\mathbb{R}^\ell$ with reflection group W. Recall that $S'_\mathcal{A}$ is the simplicial decomposition of $S^{\ell-1}$ determined by $\mathcal{A}$. Also recall that in Chapter 3, we proved that $S'_\mathcal{A}$ can be constructed to be W-invariant.

The action of W on $\mathcal{A}$ determines a group action on $\mathcal{A}_\mathbb{C}$, the complexified version of $\mathcal{A}$. To be more precise, let $z = x + iy \in \mathbb{C}^\ell$, then each $s \in W$ acts on z via

$$s(z) = s(x + iy) = s(x) + is(y).$$

Now, let $(F, C) \in \mathrm{Sal}(\mathcal{A})$. Then each $s \in W$ acts on $z(F, C)$ via

$$s(z(F, C)) = s(x(F)) + is(x(C)).$$

Since $S'_\mathcal{A}$ is W-invariant and W acts as above, the vertex set of $|\mathrm{Sal}'(\mathcal{A})|$ is W-invariant.

Since W preserves the order of the faces of $\mathrm{Sal}(\mathcal{A})$ and each $s \in W$ is linear, it is clear that each $s \in W$ will map a simplex to a simplex. This implies that $|\mathrm{Sal}'(\mathcal{A})|$ is also W-invariant.

Using Theorem 5.11, one can construct a homotopy

$$H : M(\mathcal{A}) \times [0, 1] \longrightarrow M(\mathcal{A})$$

between the identity map on $M(\mathcal{A})$ and a retraction onto $|\mathrm{Sal}'(\mathcal{A})|$. In [11], M. Salvetti constructed such an H using a collection of straight-line homotopies. So, we can choose H to be W-*equivariant* on $M(\mathcal{A})$, since W acts linearly. That is, we can choose H such that

$$H(s(z)) = s(H(z))$$

for any $s \in W$ and $z \in M(\mathcal{A})$. This implies that H induces a well-defined homotopy

$$\hat{H} : M(\mathcal{A})/W \times [0, 1] \longrightarrow M(\mathcal{A})/W$$

from the identity map on $M(\mathcal{A})/W$ to a retraction onto $|\mathrm{Sal}'(\mathcal{A})|/W$. Thus, we have the following result.

Theorem 6.1. $M(\mathcal{A})/W$ *and* $|\mathrm{Sal}'(\mathcal{A})|/W$ *are homotopy equivalent.* $\qquad\square$

The next corollary follows from Proposition 1.16.

Corollary 6.2. $M(\mathcal{A})/W$ and $|\operatorname{Sal}(\mathcal{A})|/W$ are homotopy equivalent. $\qquad\square$

Now, we will discuss how this relates to the braid arrangement. Recall that the finite reflection group corresponding to $\mathcal{A}_{\ell-1}$ is S_ℓ. Also, recall that $M(\mathcal{A}_{\ell-1})$ is homotopy equivalent to $\hat{F}_\ell(C)$. Clearly, S_ℓ acts on $\hat{F}_\ell(C)$ by permuting the labels in each point configuration. That is, all labelled configurations of the same ℓ points are in the same orbit under the action of S_ℓ. In other words, S_ℓ "removes" the labels from the point configurations. So, all of the labelled configurations from Figure 3.5 are in the same orbit under the action of S_3.

Definition 6.3. Define $F_\ell(\mathbb{C})$ to be the space of unlabelled configurations of ℓ distinct points in $\mathbb{C}$.

Example 6.4. Figure 6.1 shows two examples of unlabelled configurations of $F_3(\mathbb{C})$.
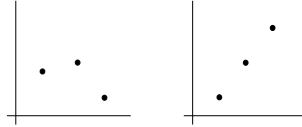

Figure 6.1: Unlabelled point configurations of $F_3(\mathbb{C})$

Proposition 6.5. $\hat{F}_\ell(\mathbb{C})/S_\ell$ is homeomorphic to $F_\ell(\mathbb{C})$. $\qquad\square$

The following corollary follows immediately.

Corollary 6.6. $M(\mathcal{A}_{\ell-1})/S_\ell$ is homeomorphic to $F_\ell(\mathbb{C})$. $\qquad\square$

The above corollary in conjuction with Corollary 6.2 imply the following.

Corollary 6.7. $F_\ell(\mathbb{C})$ and $|\operatorname{Sal}(\mathcal{A}_{\ell-1})|/S_\ell$ are homotopy equivalent. $\qquad\square$

That is, the Salvetti complex of the braid arrangement in $\mathbb{R}^\ell$ modulo the action of the symmetric group is a cell complex which is homotopy equivalent to the space of unlabelled configurations of ℓ distinct points.

Now, we wish to describe the construction of $|\operatorname{Sal}(\mathcal{A})|/W$ for all arrangements in $\mathbb{R}^2$ which have finite reflection groups. The construction that we give can be generalized for higher dimensions. For a complete proof of the general construction, see [10].

Let $\mathcal{A}$ be an essential reflection arrangement in $\mathbb{R}^2$ with reflection group W. Then we form the *dual complex* to $\mathcal{A}$, denoted $D(\mathcal{A})$, in the following way. Fix a base chamber C_0. Pick a point $x(C_0) \in C_0$. For every other chamber C of $\mathcal{A}$ pick $x(C) \in C$ so that the resulting set of points is invariant under the action of W. We can do this in the same way that we constructed S_A to be W-invariant. These points will be the vertices of $D(\mathcal{A})$. Now, join a pair of vertices $x(C_i)$ and $x(C_j)$ by an edge exactly when C_i and C_j share a common wall. This is the edge dual to the face of $\mathcal{A}$ which it crosses. Finally, form the 2-cell bounded by the polygon of dual edges. This is the 2-cell dual to $\{0\}$. It should be clear that if we construct $D(\mathcal{A})$ in this way, then $D(\mathcal{A})$ is W-invariant.

Example 6.8. Figure 6.2 depicts the dual complex of $\mathcal{A}_2$. Recall that this picture of $\mathcal{A}_2$ is a projection onto the plane defined by

$$x_1 + x_2 + x_3 = 0$$

perpendicular to the line
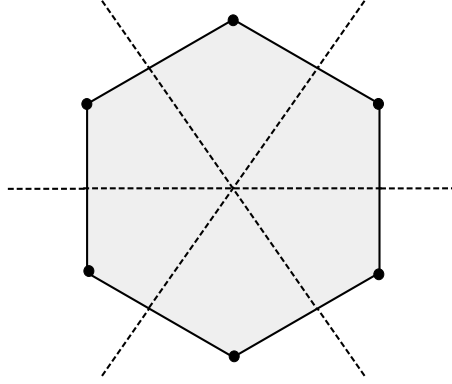
$$\{x_1 = x_2 = x_3\}.$$

Figure 6.2: Dual complex for $\mathcal{A}_2$

Now, we form the cell complex $|\operatorname{Sal}(\mathcal{A})|/W$ from $D(\mathcal{A})$ in the following way:

(i) Identify all vertices of $D(\mathcal{A})$.

(ii) Identify the edges dual to the faces F and F' exactly when F and F' are in the same orbit under the action of W.

The resulting (non-regular) cell complex is homeomorphic to $|\operatorname{Sal}(\mathcal{A})|/W$ and is homotopy equivalent to $M(\mathcal{A})/W$.

Example 6.9. Again, consider $D(\mathcal{A}_2)$. If we label the two edges dual to F_1 and F_2 connected to $x(C_0)$ as a and b, respectively, as in Figure 6.3, then we see that there are two orbits of edges: the a-type edges and the b-type edges. This is consistent with the fact that the walls of the base chamber generate the reflection group. If we identify all of the vertices and the edges based on type, then we will get a cell complex with one vertex and two edges. See Figure 6.4. The resulting orbit complex will also have one 2-cell bounded by the two edges. In this special case, the resulting cell complex is homotopy equivalent to $F_3(\mathbb{C})$.

We now want to consider an orientation of the edges of $D(\mathcal{A}_2)$. If F is a codimension 1 face of $\mathcal{A}_2$, then F is a face of exactly two chambers, say C and C'. So, (F, C) and (F, C') are both elements of $\operatorname{Sal}(\mathcal{A}_2)$. Also, $|(F, C)|$ and $|(F, C')|$ are both edges of $|\operatorname{Sal}(\mathcal{A}_2)|$ with $|(C, C)|$ and $|(C', C')|$ as their vertices. We orient each edge $|(F, C)|$ of $|\operatorname{Sal}(\mathcal{A})|$ so that the edge points toward the vertex $|(C, C)|$. It is easily verified that each 2-cell of $|\operatorname{Sal}(\mathcal{A}_2)|$ is bounded by six edges. We claim that

the orientation of the edges of $D(\mathcal{A}_2)$ given in Figure 6.3 is the same as that of the 2-cell $|(\{0\}, C_0)|$. This orientation determines the relation

$$aba = bab.$$

We could have chosen any of the 2-cells of $|\mathrm{Sal}(\mathcal{A}_2)|$ to determine the orientation of the edges of $D(\mathcal{A}_2)$. All of the 2-cells would determine the same relation given by $|(\{0\}, C_0)|$. The relation $aba = bab$ is the relation in the presentation for the fundamental group of the orbit complex $|\mathrm{Sal}(\mathcal{A}_2)|/S_3$. It is also the relation in the presentation for the braid group on three strands, which is isomorphic to the fundamental group of the unlabelled configuration space $F_3(\mathbb{C})$. Figure 6.5 shows two braids, each on three strands. The braid labelled a corresponds to the edge a of $D(\mathcal{A}_2)$. Likewise, the braid labelled b corresponds to the edge b of $D(\mathcal{A}_2)$. Then Figure 6.6 captures the identity $aba = bab$.
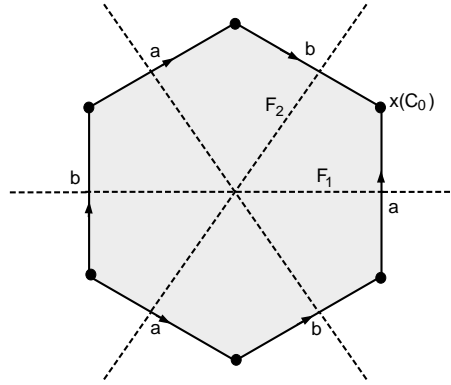

Figure 6.3: Construction of $|\mathrm{Sal}(\mathcal{A}_2)|/S_3$ from $D(\mathcal{A}_2)$
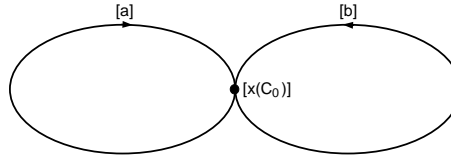

Figure 6.4: The orbit complex $|\mathrm{Sal}(\mathcal{A}_2)|/S_3$

Figure 6.5: Braids on three strands

Now, let us provide an indication as to why this construction works. First, note that if C is a chamber of $\mathcal{A}$, then $|(C, C)|$ and $|(\{0\}, C)|$ are both cells of $|\mathrm{Sal}(\mathcal{A})|$. Also, recall that cells of the
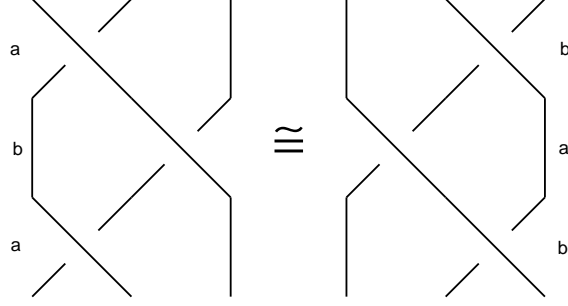
Figure 6.6: The identity $aba = bab$

form $|(C, C)|$ are the vertices of $|\operatorname{Sal}(\mathcal{A})|$. And cells of the form $|(\{0\}, C)|$ are the top dimensional cells of $|\operatorname{Sal}(\mathcal{A})|$. In the case where $\mathcal{A}$ is an arrangement in $\mathbb{R}^2$, the top dimensional cells are 2-cells.

The action of W on $|\operatorname{Sal}(\mathcal{A})|$ is given by

$$w|(F, C)| = |(wF, wC)|,$$

where $w \in W$. So, W acts on the vertices of $|\operatorname{Sal}(\mathcal{A})|$ in the same way that W acts on the vertices of $|\operatorname{Sal}'(\mathcal{A})|$. That is, $w \in W$ acts on $|(C, C)| \in |\operatorname{Sal}(\mathcal{A})|$ via

$$w(|(C, C)|) = w(x(C)) + iw(x(C)).$$

Since W acts transitively on chambers, it follows that there is one orbit of cells of type $|(C, C)|$. Also, since $w\{0\} = \{0\}$, there is one orbit of cells of type $|(\{0\}, C)|$. Thus, $|\operatorname{Sal}(\mathcal{A})|/W$ has one vertex and one top dimensional cell.

For a 2-dimensional arrangement $\mathcal{A}$, we can think of each vertex of $D(\mathcal{A})$ as a vertex of $|\operatorname{Sal}(\mathcal{A})|$, which is of the type $|(C, C)|$, and the 2-cell of $D(\mathcal{A})$ as the orbit of the top dimensional cells of $|\operatorname{Sal}(\mathcal{A})|$, which are of the type $|(\{0\}, C)|$.

Let F be a 1-codimensional face of an arrangement $\mathcal{A}$ in $\mathbb{R}^2$. Then F is a face of exactly two chambers, say C and C'. So, (F, C) and (F, C') are both elements of $\operatorname{Sal}(\mathcal{A})$. We claim that $|(F, C)|$ and $|(F, C')|$ are in the same orbit under the action of the reflection group. Indeed, reflection across the hyperplane containing F will carry $|(F, C)|$ to $|(F, C')|$. We can think of each edge in $D(\mathcal{A})$ as a pair of edges of $|\operatorname{Sal}(\mathcal{A})|$ of the type described above. That is, the edge of $D(\mathcal{A})$ that is dual to the face F represents the identified edges $|(F, C)|$ and $|(F, C')|$ of $|\operatorname{Sal}(\mathcal{A})|$.

For 2-dimensional arrangements, we can think of $D(\mathcal{A})$ as a "partially identified" $|\operatorname{Sal}(\mathcal{A})|$, where all of the top-dimensional cells have been identified and some of the edges have been identified. The remaining identifications of edges of $|\operatorname{Sal}(\mathcal{A})|$ can be shown to arise from identifications of the edges of $D(\mathcal{A})$ arising from the action of W as described above.

Let $\mathcal{A}$ be a reflection arrangement with group W. Fix a chamber C_0. Let $H_1, \ldots, H_n$ be the walls of C_0. Then W is generated by the reflections $s_{H_1}, \ldots, s_{H_n}$, according to Lemma 3.7. Also, according to [7], W has presentation

$$\langle s_{H_1}, \ldots, s_{H_n} : (s_{H_i})^2 = 1, s_{H_i} s_{H_j} s_{H_i} \cdots s_{H_i} = s_{H_j} s_{H_i} s_{H_j} \cdots s_{H_j} \rangle$$

where m_{ij} factors appear on each side of the second relation for some integers m_{ij} with $i \neq j$. The m_{ij} are easily determined by the "Coxeter graph" associated with the type of W.

The following well-known result given in [4] is easy to see from the description of $|\operatorname{Sal}(\mathcal{A})|/W$ in terms of the dual complex. Example 6.9 is the special case with W of type A_2.

Theorem 6.10. *Let $\mathcal{A}$ be a reflection arrangement with group W. Then $\pi_1(M(\mathcal{A})/W)$ has presentation*

$$\langle g_1, \ldots, g_n : g_i g_j g_i \cdots g_i = g_j g_i g_j \cdots g_j \rangle$$

with m_{ij} factors on each side of the relation. $\qquad\square$

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